PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 33
Perturbation theory: many degrees of freedom

November 13

Today: perturbation theory from old AA (ϕ_{0A}, J_A)

$$\downarrow$$

new AA (ϕ_A, J_A)

given Hamiltonian $H = H_0(\vec{J}_0) + \varepsilon H_1(\vec{J}_0, \vec{\phi}_0)$

In new coords: want $H = H(\vec{J}) \rightarrow$ integrable!

Follow lec32: look for Type 2 CT

$$S(\vec{\phi}_0, \vec{J}) = \underbrace{\vec{\phi}_0 \cdot \vec{J}}_{\sum_A \phi_{0A} J_A} + \varepsilon S_1(\vec{\phi}_0 + \vec{J}) + \text{higher orders}\ldots$$

$$\downarrow$$

$$\phi_A = \frac{\partial S}{\partial J_A} = \phi_{0A} + \varepsilon \frac{\partial S_1}{\partial J_A} + \cdots$$

$$J_{0A} = \frac{\partial S}{\partial \phi_{0A}} = J_A + \varepsilon \frac{\partial S_1}{\partial \phi_{0A}} + \cdots$$

In new coords:

$$H = H_0(\vec{J}_0) + \varepsilon H_1(J_{0A}, \phi_{0A})$$

$$H_0(\vec{J}_0) \to H_0\left(J_A + \varepsilon \frac{\partial S_1}{\partial \phi_{0A}}\right) \qquad \varepsilon H_1(J_{0A}, \phi_{0A}) \to \varepsilon H_1(J_A, \phi_{0A}) + \cdots$$

$$H = H_0(J_A) + \varepsilon \left[\sum_A \underbrace{\frac{\partial H_0}{\partial J_A}}_{=\omega_{0A}} \frac{\partial S_1}{\partial \phi_{0A}} + H_1(J_A, \phi_{0A}) \right]$$

Expand H_1, S_1 as Fourier series:

$$S_1 = \sum_{\text{each } m_A = -\infty}^{\infty} e^{i\vec{m}\cdot\vec{\phi}_0} s_{\vec{m}}(\vec{J}) \qquad H_1 = \sum_{m_A=-\infty}^{\infty} e^{i\vec{m}\cdot\vec{\phi}_0} h_{\vec{m}}(\vec{J})$$

We need:
$$\varepsilon \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}_0} \left[\sum_A \omega_{0A}\, i m_A\, s_{\vec{m}}(\vec{J}) + h_{\vec{m}}(\vec{J}) \right]$$

choose given

to be ind. of $\vec{\phi}_0$.

Choose: $s_{\vec{0}}(\vec{J}) = 0.$ (if all $m_A = 0$)

$$s_{\vec{m}} = -\frac{h_{\vec{m}}(\vec{J})}{\underbrace{i\vec{m}\cdot\vec{\omega}_0}}\quad (\text{if } \vec{m} \neq \vec{0})$$

And:
$$H = H_0(\vec{J}) + \varepsilon h_{\vec{0}}(\vec{J}) + 0 = H(\vec{J})$$

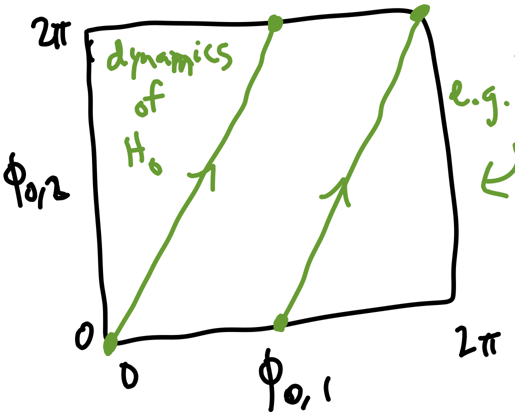
problem stayed integrable?

Any loophole? $\vec{m}\cdot\vec{\omega}_0 = 0$?

if happens when $h_{\vec{m}} \neq 0$, then $\underbrace{\text{PT fails!}}$

regions where AA defined "split"

Suppose we have 2 DoF. Recall:



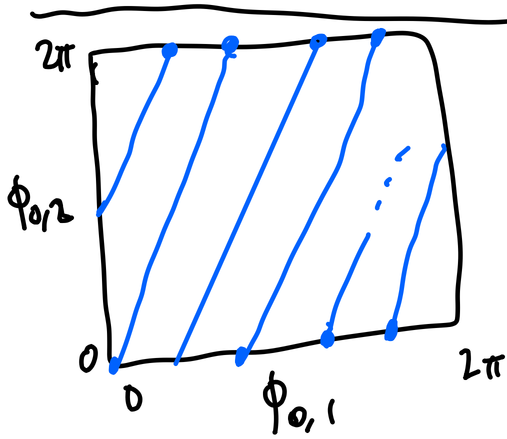
commensurate frequencies: $\vec{\omega}_0 = (\omega_1, \omega_2)$

e.g. $\frac{\omega_1}{\omega_2} = \frac{1}{2}$

where $\frac{\omega_1}{\omega_2}$ = rational = $\frac{\text{integer}_1}{\text{integer}_2}$

here PT fail if:

- $m_1 \omega_1 + m_2 \omega_2 = 0$ $\quad m_1 + 2m_2 = 0$
- e.g. $m_1 = 2$, $m_2 = -1$ need: $h_{2,-1}(\vec{J}) \neq 0$.



incommensurate frequencies: $\frac{\omega_1}{\omega_2}$ = irrational

Trajectories are <u>dense</u> on T^2.

- explore 2d of phase space.

Strictly: no $\vec{m}$ where $\vec{m} \cdot \vec{\omega}_0 = 0$.

<u>but</u>: $\vec{m} \cdot \vec{\omega}_0$ can be as small as wanted?

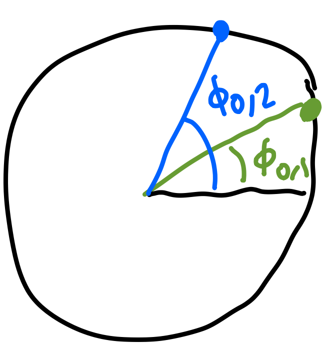
If: $\varepsilon \frac{h_{\vec{m}}}{i\vec{m} \cdot \vec{\omega}_0}$ small ... then $h_{\vec{m}}$ need to decay fast enough.

(KAM: Lec. 42 – it is possible for $h_{\vec{m}} \to 0$ fast enough...)

Conclusion: $n > 1$ DOF: PT can fail

→ commensurate orbits vulnerable ⇝ chaos

Example: 2 particles on ring.



$$H = \underbrace{\left[\frac{J_{0,1}^2}{2I} + \frac{J_{0,2}^2}{2I} \right]}_{H_0} + \varepsilon \cos(\phi_{0,1} - \phi_{0,2})$$

First order PT:

$$h_{m_1, m_2} = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi_{0,1} \int_0^{2\pi} d\phi_{0,2} \; H_1 \, e^{-i(m_1 \phi_{0,1} + m_2 \phi_{0,2})}$$

Or: $\cos(\phi_{0,1}-\phi_{0,2}) = \frac{1}{2}\left[e^{i(\phi_{0,1}-\phi_{0,2})} + e^{-i(\phi_{0,1}-\phi_{0,2})}\right]$

which $h_{\vec{m}} \neq 0$? $(m_1, m_2) = (1,-1)$ $= (-1, 1)$

$h_{1,-1} = h_{-1,1} = \frac{1}{2}$.

$$s_{1,-1} = -\frac{1}{2i(\omega_{0,1}-\omega_{0,2})} = -s_{-1,1}$$

$$S_1 = \phi_{0,1}J_1 + \phi_{0,2}J_2 + \frac{\sin(\phi_{0,1}-\phi_{0,2})}{2(\omega_{0,1}-\omega_{0,2})}$$

Can PT fail?

$$\omega_{0,1} = \frac{\partial H_0}{\partial J_1} = \frac{J_1}{I} \quad \text{and} \quad \omega_{0,2} = \frac{\partial H_0}{\partial J_2} = \frac{J_2}{I}$$

Perturbation theory fails if $J_1 \approx J_2$.

To understand why, switch coords: (via CT)

$$\bar{J} = J_{0,1} + J_{0,2} \qquad \bar{\phi} = \frac{\phi_{0,1}+\phi_{0,2}}{2}$$

$$\tilde{J} = \frac{J_{0,1}-J_{0,2}}{2} \qquad \tilde{\phi} = \phi_{0,1}-\phi_{0,2}$$

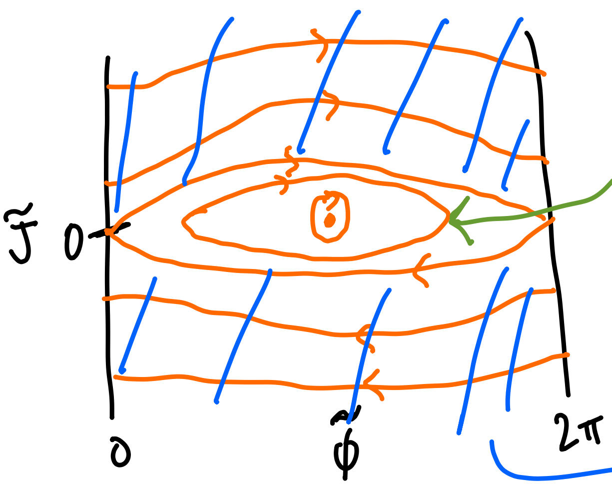
$\{\bar{\phi}, \bar{J}\} = 1$

$\{\tilde{\phi}, \tilde{J}\} = 1$.

"center of mass coords"

$$\hookrightarrow H = \frac{\bar{J}^2}{4I} + \frac{\tilde{J}^2}{I} + \varepsilon\cos\tilde{\phi}$$

[exact, not PT]



When $\tilde{J} \sim \sqrt{\varepsilon I}$, PT fails.

shape of orbit changed from

$(\tilde{\phi}, \tilde{J})$

AA vars smoothly connected to $(\tilde{\phi}, \tilde{J})$