# PHYS 5210
# Graduate Classical Mechanics
# Fall 2023

## Lecture 34
### Hénon-Heiles Hamiltonian

November 14

lec 31: integrable system is exactly solvable
↳ find AA coords $(\phi_A, J_A)$

lec 33: sometimes, perturbations destroy AA coords?
① Do we just get new AA coords? still integrable...
② or no AA coords exist? ⟶ chaos
↓
observe today!

How to observe chaos in Hamiltonian system? [H t-ind.]

Recall: system on 2n-dim phase space
= integrable if we find n ind. consts. $\{J_A, J_B\} = 0$.

If $n=1$? $J_1 = H$. we found $n=1$ const. of motion

ALL 1-DOF systems integrable.

If $n=2$? Yes, we can find chaos!

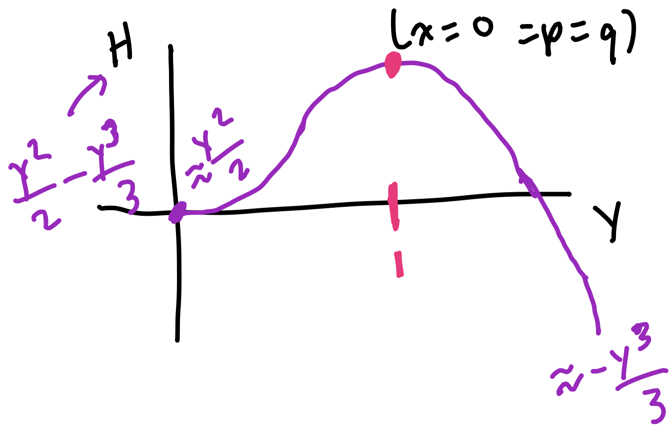
Explicitly via example:

$\left[\begin{array}{c}\text{Hénon-Heiles} \\ \text{1970s}\end{array}\right]$: $H = \dfrac{p_x^2+p_y^2}{2m} + \dfrac{1}{2}k(x^2+y^2) + b\left(x^2y - \dfrac{y^3}{3}\right)$

Remove clutter ($m=k=b=1$): $p_x \to p$, $p_y \to q$

$$H = \frac{p^2+q^2}{2} + \frac{x^2+y^2}{2} + x^2y - \frac{y^3}{3}.$$

Dynamics has Hamiltonian $H$, $t$-ind. ... so $H$ conserved.




$\dfrac{\partial H}{\partial y} = 0 = y - y^2 = 0$

$\hookrightarrow$ $y=1$

When $y=1$, $x=p=q=0$,

$H = 1/6$.

Restrict $H \leq 1/6$. $\rightarrow$ trajectories stay bounded.

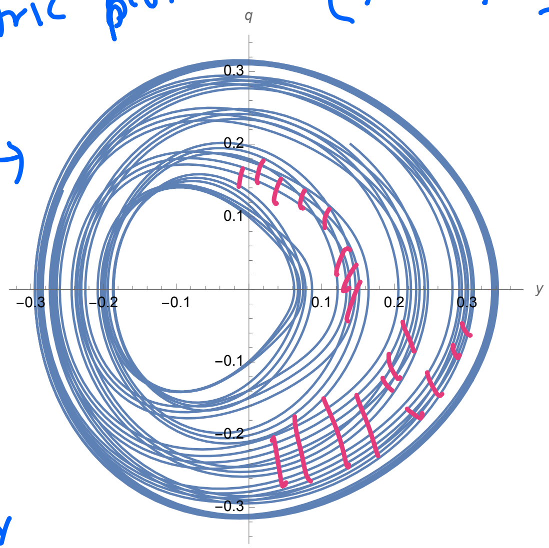
Hamilton's equations?

$$\left.\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p} = p \\
\dot{y} &= \frac{\partial H}{\partial q} = q \\
\dot{p} &= -\frac{\partial H}{\partial x} = -x - 2xy \\
\dot{q} &= -\frac{\partial H}{\partial y} = -y - x^2 + y^2.
\end{aligned}\right\}$$

Plug into Mathematica...

Here's a simulation of $H \approx 0.058$

Parametric plot of $[y(t), q(t)]$ : projection from 4D phase space $\to$ 2D.

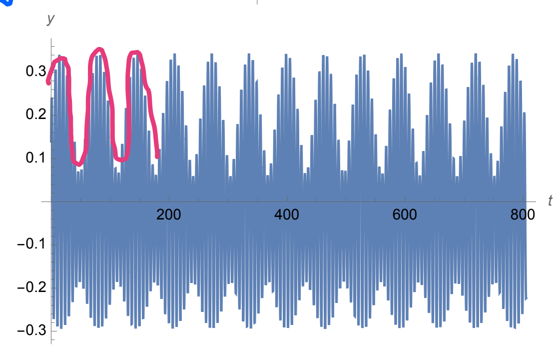



If dynamics was integrable: there'd be AA variables... motion on 2d torus $(\phi_1, \phi_2)$

If $\omega_1, \omega_2$ commensurate: trace 1d curve.

$\boxed{\omega_1/\omega_2 \text{ incommen., fill in 2d}}$

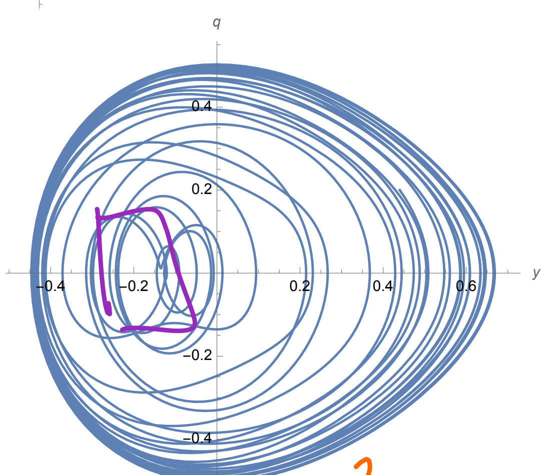
$y(t)$ alone.



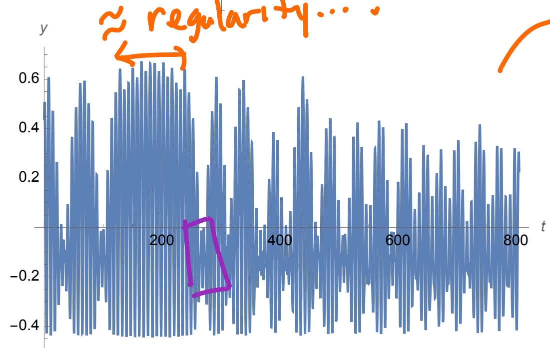

dynamics ~ quasiperiodic.

numerics $\to$ integrability (not proof...)

Increase $H \approx 0.13 < 1/6$.




≈ regularity...?




irregular!

not quasiperiodic? (not proof)

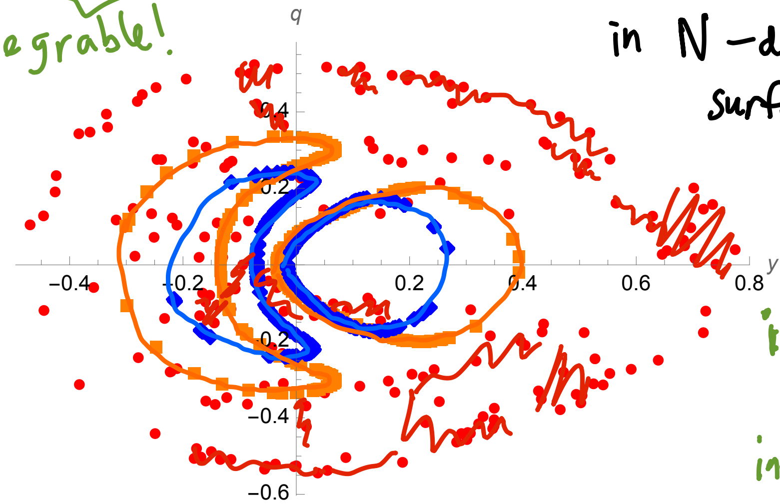
$\downarrow$

Chaos?

Observe: Poincaré section. Fix H. Record values of $(y, q)$ when $x=0$.

$E \approx 0.03$ $\quad$ $E \approx 0.06$ $\quad$ $E \approx 0.15 \leftarrow$ chaos!

$\underbrace{E \approx 0.03 \quad E \approx 0.06}_{\text{integrable!}}$

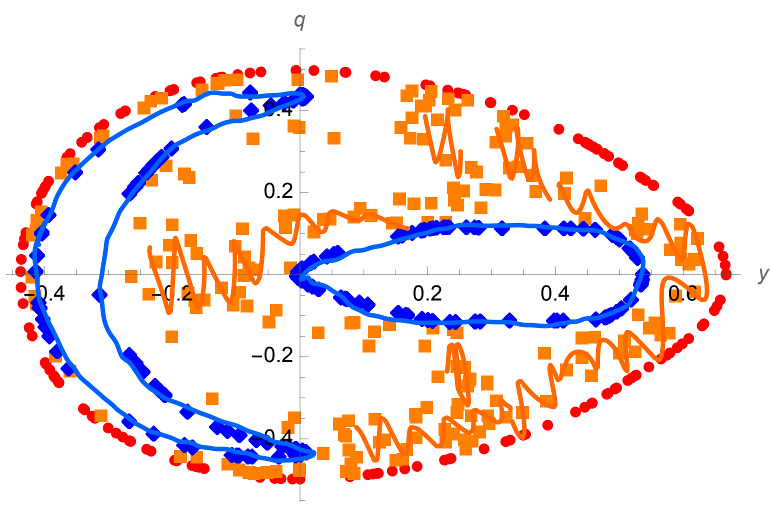



in N-dim phase space...
surface $S_1$ w/ dim $N-n_1$
$S_2$ ... $N-n_2$
intersect on $N-n_1-n_2$ dim.

integrable? $\underset{4-1d}{S_1=\{x=0\}}$; $\underset{4-2d}{S_2=T^2_{\phi_1\phi_2}}$

intersect on $4-2-1=1$-dim surf.

chaos? $\underset{4-1d}{S_1=\{x=0\}}$ & $\underset{4-1d?}{S_2=\{\text{const. } H\}}$? $\rightarrow$ intersect on 2d? ($>1$ d).




Also Poincaré section, $x=0$.

But, $E=1/8$ fixed.

diff color $\rightarrow$ diff ICs.

Some ICs $\rightarrow$ integrable?

other ICs $\rightarrow$ chaos?

These appear for same E! Coexistence of chaos & integrability!