PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 35
Kicked rotor

November 17

lec 34: t-ind $H \to$ 2 DOF for chaos

Today: t-dep $H \to$ 1 DOF for chaos

Example: kicked rotor. let $\{\phi, J\} = 1$. $\underbrace{\phi \sim \phi + 2\pi}_{\text{periodically identify}}$

$$H(t) = \frac{J^2}{2I} + \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \cdot \varepsilon_0 \cos\phi$$

Dirac δ function.

EoMs:

$$\dot{\phi} = \frac{\partial H}{\partial J} = \frac{J}{I}$$

$$\dot{J} = -\frac{\partial H}{\partial \phi} = \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - n\tau)\, \varepsilon_0 \sin\phi}$$

$\dot{J} = 0$ if $t \neq n\tau$.

So $J(t)$ will be piecewise const.

Let's suppose at $t=0^+$ (just after $t=0$):

$\phi(0^+) = \phi_0$, $J(0^+) = J_1 \leftarrow$ explain later!

Since $\dot{J}=0$ for $0<t<\tau \longrightarrow J(t) = J_1 \quad (0<t<\tau)$.

And $\dot{\phi} = \frac{J}{I}$ means $\phi(t) = \phi_0 + \frac{J_1}{I} t \quad (0<t<\tau)$.

Just before $t=\tau$ kick: $\phi(\tau^-) = \phi_1 = \phi_0 + \frac{\tau}{I} J_1$ ①

What about after kick?

$$J(\tau+\delta) = J(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt\, \dot{J}(t) = J(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt \sum_{n=-\infty}^{\infty} \delta(t-n\tau)\, \varepsilon_0 \sin\phi(t)$$

$$\int_{\tau-\delta}^{\tau+\delta} dt\, \delta(t-\tau) f(t) = f(\tau)$$

$$= J_1 + \varepsilon_0 \sin\phi(\tau) \quad ②$$

[other $n\neq 1$ term $\to 0$]

$$J(\tau^+) = J_2 = J_1 + \varepsilon_0 \sin\phi_1$$

Justify $\phi(\tau) = \phi_1$?

$$\phi(\tau+\delta) = \phi(\tau-\delta) + \int_{\tau-\delta}^{\tau+\delta} dt\, \dot{\phi} = \phi(\tau-\delta) + \cancel{\int_{\tau-\delta}^{\tau+\delta} dt}\, \frac{J(t)}{I},$$

$\leq \delta \cdot \frac{1}{I} \max(J_1, J_2) \to 0$.

So $\phi(t)$ is continuous

Iterate this argument, define

$\phi(n\tau^-) = \phi_n$, $J(n\tau^-) = J_n$:

$$\left.\begin{aligned} ② &\rightarrow \quad J_{n+1} = J_n + \varepsilon_0 \sin\phi_n \\ ① &\rightarrow \quad \phi_{n+1} = \phi_n + \frac{\tau}{I} J_{n+1} \end{aligned}\right\}$$

reduces Hamilton's eqs (ODEs) to discrete map.

Reduced to math... work in dimensionless units!

$[\phi] = $ const.

$[t] = [T]$ time

$[J] = [M][L]^2[T]^{-1}$ (mass, length)

$[\tau] = [T]$

$[\varepsilon_0] = [J] = [M][L]^2[T]^{-1}$

$[I] = [M][L]^2$

Define $\tilde{\phi}_n = \phi_n$ (dim-less), $\tilde{J}_n = \dfrac{J_n}{I/\tau}$. $\varepsilon = \dfrac{\varepsilon_0 \tau}{I}$ dimensionless kick strength

Dimensionless kicked rotor:

$$\left.\begin{aligned} \tilde{J}_{n+1} &= \tilde{J}_n + \varepsilon \sin \tilde{\phi}_n \\ \tilde{\phi}_{n+1} &= \tilde{\phi}_n + \tilde{J}_{n+1} \end{aligned}\right\}$$

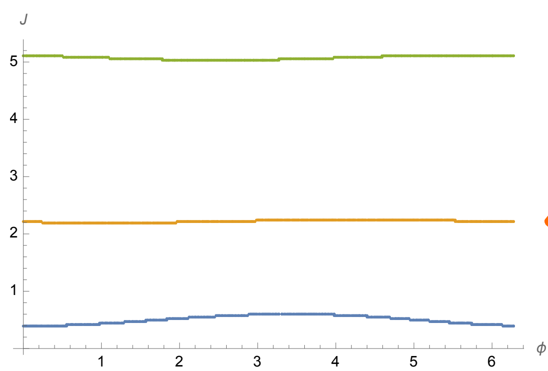
NOTE: $\tilde{\phi}_n + 2\pi \to \tilde{\phi}_n$ (same pt in phase space)

$\tilde{J}_n + 2\pi \to \tilde{J}_n$ doesn't change any $\tilde{\phi}_n$...

going to identify $\tilde{J} \sim \tilde{J} + 2\pi$.

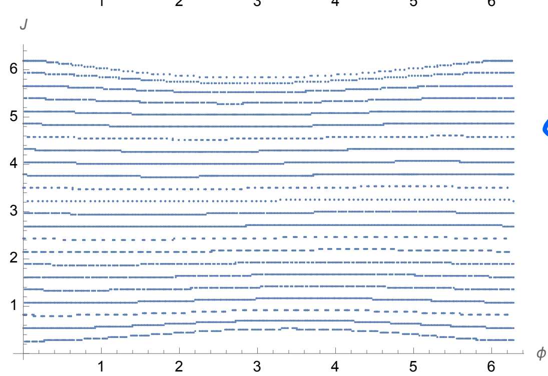
Henceforth, drop tildes...

Numerical simulations: $\varepsilon = 0.05$



← plot of 3 diff. ICs (each color)

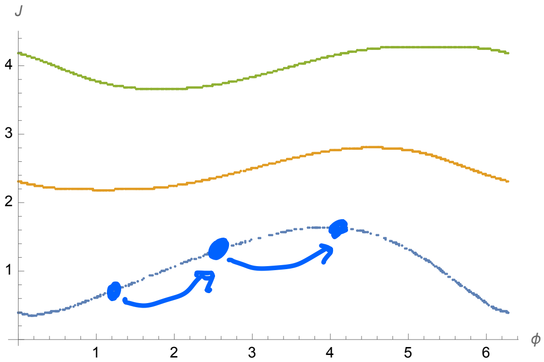
parametric plot (ϕ_n, J_n)



← 20 diff ICs.

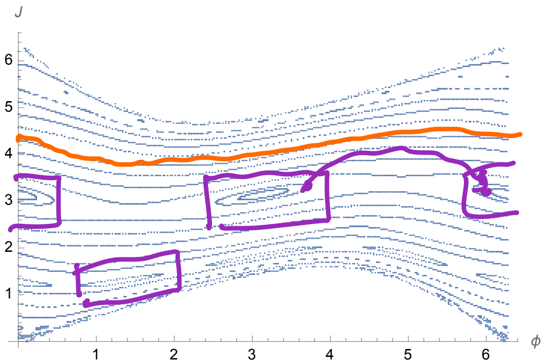
Punchline: each trajectory seems confined to 1d?

Dynamics looks integrable : motion restricted to 1d?



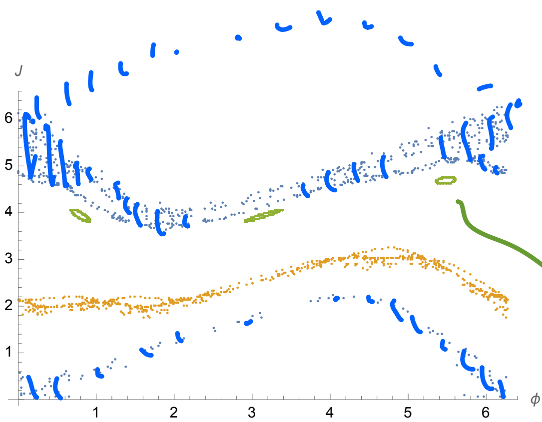
Numerics at $\varepsilon = 0.6$.

still integrable?



some "trajectories" (ϕ_n, J_n) wind around ϕ.

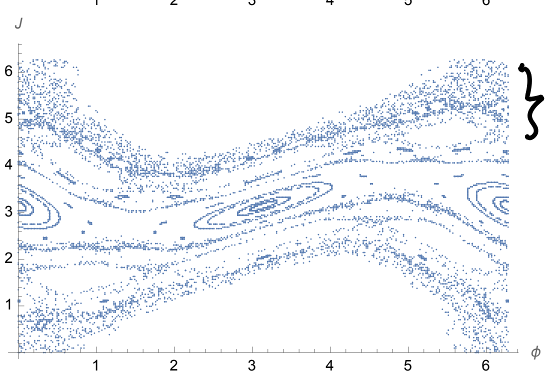
trapped "orbits" also exist.



Numerics at $\varepsilon = 1.05$.

chaotic dynamics?

this seems "integrable"?

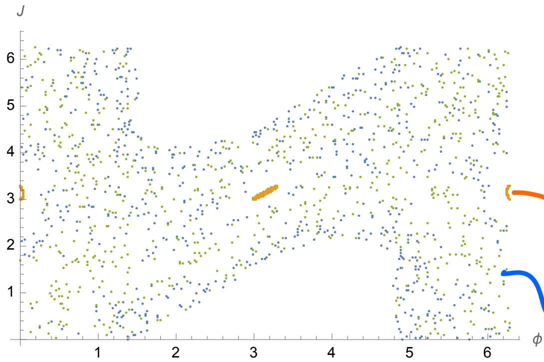


} which IC?? $\to$ chaos

"onset of chaos" at

$\varepsilon_c \sim 0.95$.

Numerics at $\varepsilon = 1.8.$

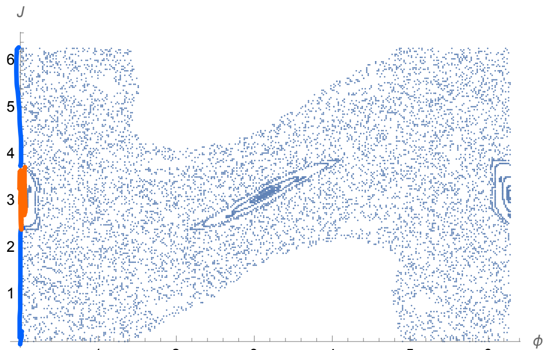


"island of integrability"

blue/green ICs "mix"

chaos

integrable



all dynamics started at $\phi_0 = 0$.