PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 36
Logistic map

November 27

Chaotic models:

① Hénon-Heiles
Hamiltonian dynamics
(t-ind.)
4D phase space

② kicked rotor
Hamiltonian
(t-dep.)
~~2D phase space~~
discrete map

③ Today:
logistic map.
discrete map
dissipative

Logistic map: $x_{n+1} = f(x_n) = r \cdot x_n(1-x_n)$.

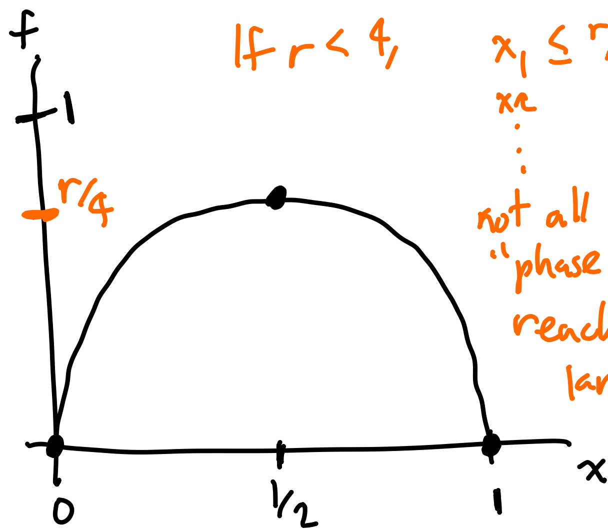
What is behavior of $\{x_0, x_1, \ldots, x_n\}$?

Goal: $0 \le x_n \underbrace{\le 1}_{\text{then } x_{n+1} < 0}$

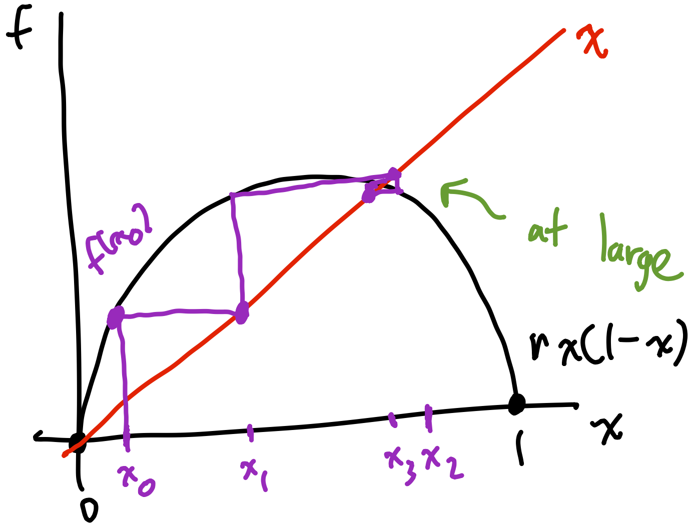
$r \ge 0$

At $x = 1/2$: $\underbrace{f(x) = r/4}_{\le 1 \text{, need } \underline{r \le 4}}$.

So: $0 \le r \le 4$, and $0 \le x_0 \le 1$.



If $r < 4$, $x_1 \le r/4$.
x_2
⋮
not all of "phase space" reached at large n.



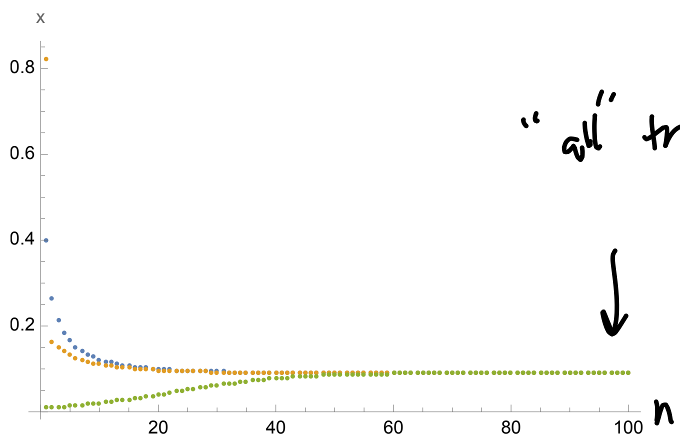
"Cobweb diagram" implements logistic map via pictures

at large n, spiral towards fixed point.
stable

As before, easiest to see numerical simulations.

Start by plotting $x_0, x_1, x_2 \ldots$ vs. n.
Each color = diff x_0.

$r = 1.1$



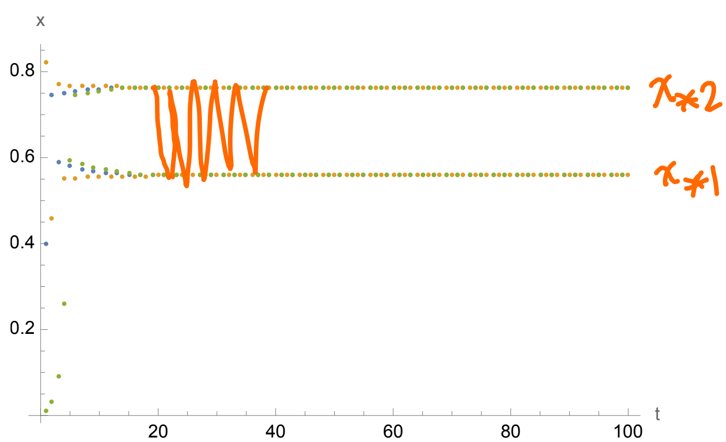
"all" trajectories approach a stable fixed pt:

$f(x_*) = x_*$

$r x_* (1 - x_*) = x_*$

$\hookrightarrow x_* = 1 - 1/r = \frac{1}{11}$.

$r = 3.1$



trajectories approach stable period-2 cycle:

$f(x_{*1}) = x_{*2}$

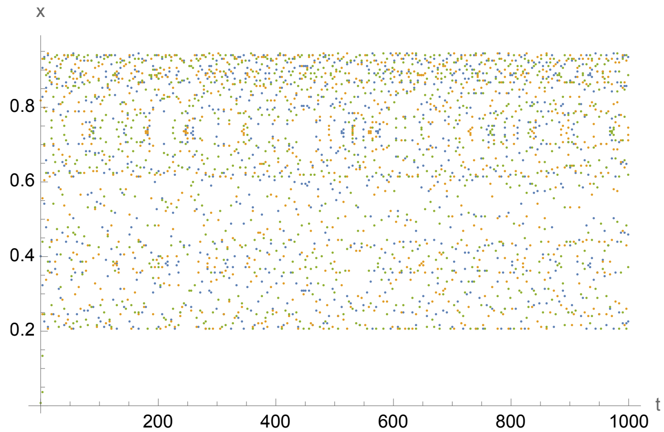
$f(x_{*2}) = x_{*1}$

Definition: period-n cycle has

$$x_{*1} \xrightarrow{f} x_{*2} \xrightarrow{f} \cdots \xrightarrow{f} x_{*n} \to x_{*1}$$

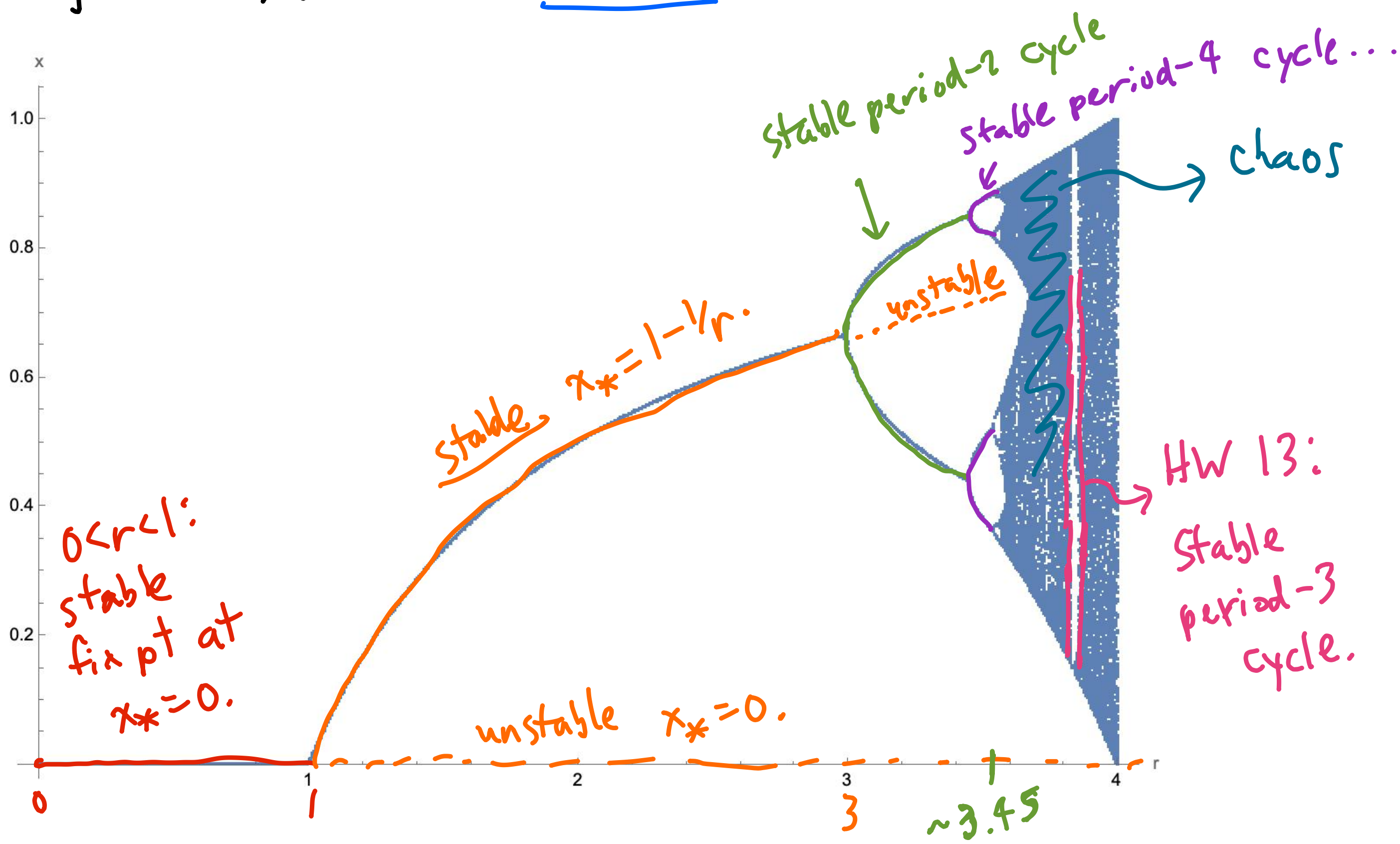
(Onset of) chaos — erratic trajectories

$r = 3.77$

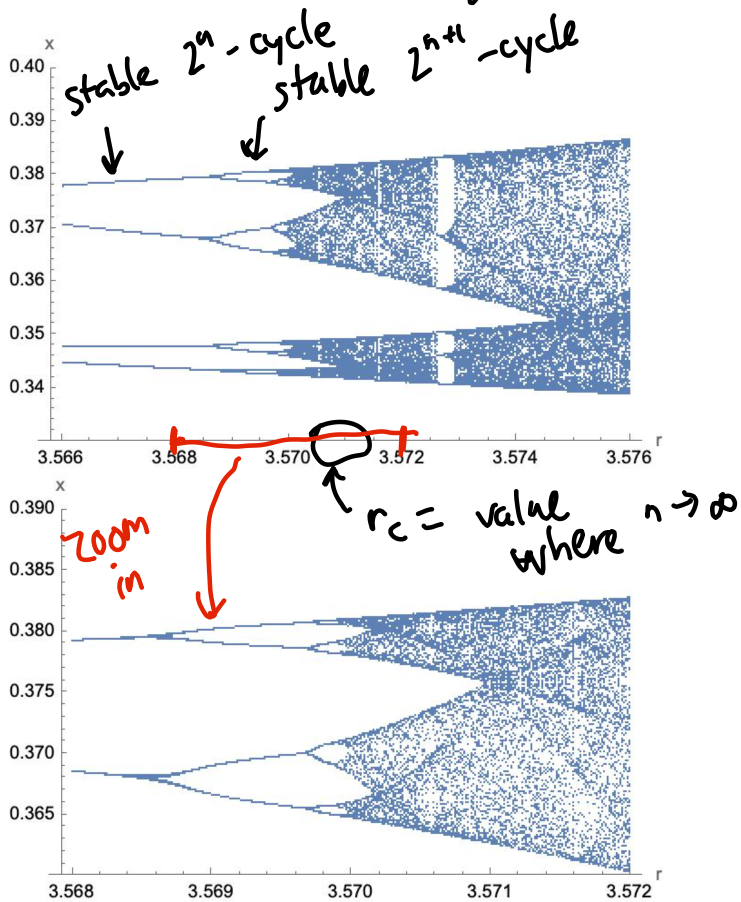


Since is dissipative, "blending" of ICs is not good diagnostic... LOT of phase space $[0,1]$ hit as $n \to \infty$.

Plot of possible late $n \to \infty$ values of x_n: for given r, set = attractor



The onset of chaos in logistic map: period doubling (lec 38)



Attractor shape near r_c (onset of chaos) looks $\underbrace{\text{self-similar...}}$

Attractor becomes a fractal-set characterized by non-integer dimension (lec 39, 40)

- Chaos arising due to period doubling ($n \to \infty$ as $r \to r_c$)
- Self-similarity of attractor → quantitatively universal, general dissipative maps have same attractor shape as $r \to r_c$. (lec 38).