PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 37
Linear stability analysis

November 29

Logistic map: $x_{n+1} = r x_n (1-x_n)$ $0 \leq x_n \leq 1$, $0 \leq r \leq 4$

Minimal model for chaos.

Today: analytic understanding for $r \lesssim 3.45$.

Claim: if $r<1$, $x_n \to 0$ as $n \to \infty$.

Proof: $\dfrac{x_{n+1}}{x_n} = \underbrace{r}_{<1} \underbrace{(1-x_n)}_{\leq 1} \leq r < 1$. so x_n decreasing...

$x_n \leq r^n$.

Approach 0 exponentially fast

$x_* = 0$ is a fixed point: $x_* = f(x_*)$ $f(x) = rx(1-x)$

$x_* = 0$ is stable fixed point.

For now: consider generic map $x_{n+1} = f(x_n)$,

fixed point $x_* = f(x_*)$.

Develop theory of linear stability near fixed point:

$x_n = x_* + \delta x_n$ infinitesimal... work to linear order.

So:

$x_{n+1} = f(x_n)$

$\cancel{x_*} + \delta x_{n+1} = f(x_* + \delta x_n) = \overbrace{\cancel{f(x_*)}}^{x_*} + \overbrace{f'(x_*)}^{\text{call this } \lambda.} \delta x_n$

$\delta x_{n+1} = \lambda \delta x_n$

$\delta x_n = \lambda^n \delta x_0$.

Key question: does

$|\delta x_n| \to \infty$ as $n \to \infty$
[infinitesimal approx. fails eventually]
x_* unstable fix pt.

$|\delta x_n| \to 0$ as $n \to \infty$
x_* stable fix pt

Since $\frac{|\delta x_n|}{|\delta x_0|} = |\lambda|^n$, so $[\lambda = f'(x_*)]$

$|\lambda| > 1$: unstable

$|\lambda| < 1$: stable

If $\lambda = \pm 1$: Fixed point marginal, need to keep expanding $f(x)$... linear analysis fails.

Return to logistic map: $f(x) = rx(1-x)$

Use linear stability theory to learn as much as possible...

① Find fixed points.

$$f(x_*) = r x_* (1-x_*) = x_* . \quad \text{Solved by:}$$

$\underline{x_* = 0}$ and $r(1-x_*) = 1$

$x_* = \underbrace{1 - 1/r}.$

valid fixed point only when $0 < 1 - 1/r < 1$

or $\underline{r > 1}$.

② Analyze stability: calculate $\lambda = f'(x) = r(1-x) - rx$
$= r(1-2x)$

$x_* = 0$:

$|\lambda| = r\,|1 - 2\cdot 0| = r$

so this:

stable if $r < 1$

unstable if $r > 1$

$\underline{\text{marginally stable}}$ at $r = 1$

$\frac{x_{n+1}}{x_n} = 1 - x_n < 1$ if $x_n > 0$

$x_* = 1 - 1/r$ $(r > 1)$:

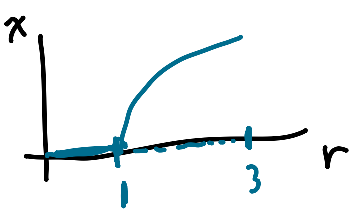
$|\lambda| = r\,|1 - 2(1 - 1/r)|$

$= r\,|\frac{2}{r} - 1| = |2 - r|$

stable if $1 < r < 3$

unstable if $r > 3$

So far:



Once $r > 3$, both fixed pts unstable...

Numerically: what's stable is period-2 cycle (lec 36)

Analytically: define n^{th} iterated map:

$$f^{[n]}(x) = f(f^{[n-1]}(x)) \quad \text{w/} \quad f^{[1]} = f.$$

Namely: $x_{\ell+n} = f^{[n]}(x_\ell)$

If period-2 cycle, then $f^{[2]}$ have fixed point.

Goal: find fixed pts of $f^{[2]}$: $\quad x_* = f^{[2]}(x_*)$.

or: $\quad x_* = r\left[r x_*(1-x_*)\right]\left[1 - r x_*(1-x_*)\right]$

Quartic equation !! ☹

We know 2 solutions: $x_* = 0$ and $x_* = 1 - 1/r$.

b/c $\quad x_* = f(x_*) = f(f(x_*))$.

$$0 = \frac{x_* - f^{[2]}(x_*)}{x_*\left[x_* - 1 + 1/r\right]} \xrightarrow{\text{Mathematical}} 0 = r^2 x_*^2 - r(1+r)x_* + (1+r)$$

Use quadratic formula:

$$x_* = \frac{r(1+r) \pm \sqrt{r^2(1+r)^2 - 4r^2(1+r)}}{2r^2} = \frac{1+r \pm \sqrt{(1+r)(r-3)}}{2r}$$

only real if $r > 3$.

infinitesimal, $\varepsilon > 0$

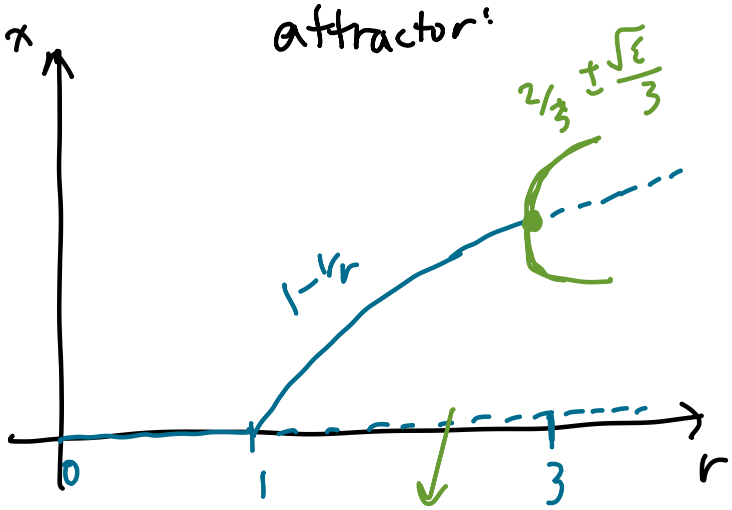
If $r = 3 + \varepsilon$:

$$\boxed{x_* \simeq \frac{2}{3} \pm \frac{\sqrt{\varepsilon}}{3}}$$

$= 1 - \frac{1}{r}$, $\quad (r=3)$

fixed pt that went unstable

attractor:



Period doubling!

Last thing: check stability of period-2 cycle.

Stable if $|\lambda| < 1 \quad \left| f^{[2]\prime}(x_*) \right|$ either point on period-2 cycle.

Mathematica: $|\lambda| = \cdots = 4 + 2r - r^2$

Mathematica!

Regions of stability: $\pm 1 = 4 + 2r - r^2$

$\downarrow$

$r^2 - 2r - (3 \text{ or } 5) = 0$

$r = \dfrac{2 \pm \sqrt{4 + 4 \cdot 3}}{2}$

$= 1 \pm 2$

$\downarrow$

$r = 3$

$r = \dfrac{2 \pm \sqrt{4 + 4 \cdot 5}}{2}$

$= 1 \pm \sqrt{6}$

$\downarrow$

$r = 1 + \sqrt{6} \approx 3.45$.

Stable period-2 cycle when

$3 < r < 3.45$.

[Check by $|\lambda| < 1$ at $r \sim 3.1$]

At $r = 3.45$ — period-2 cycle becomes unstable

$\downarrow$

Stable period-4 cycle

$\downarrow$

$\vdots$