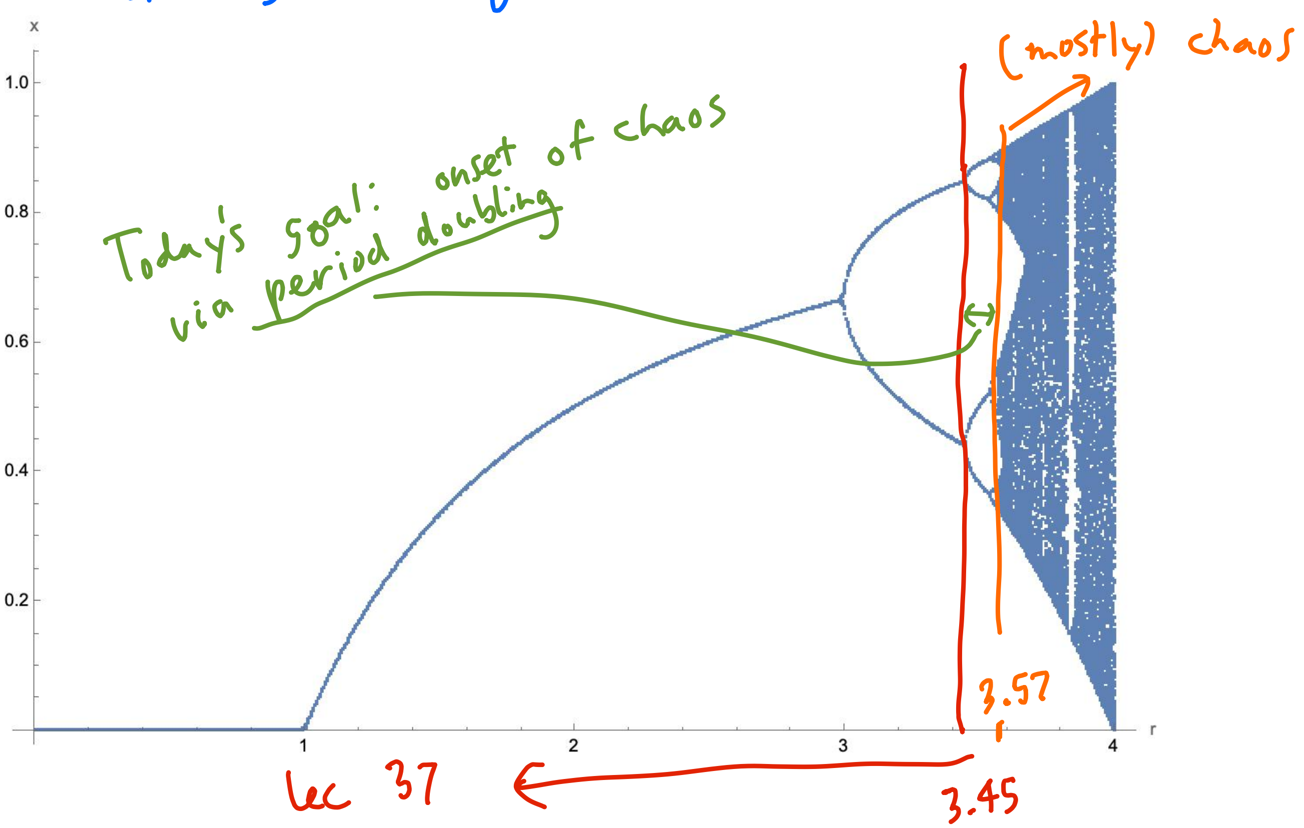
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 38
Period doubling

December 1

Logistic map: $x_{n+1} = r x_n (1 - x_n)$ $\qquad 0 \le x_n \le 1$, $\quad 0 \le r \le 4$

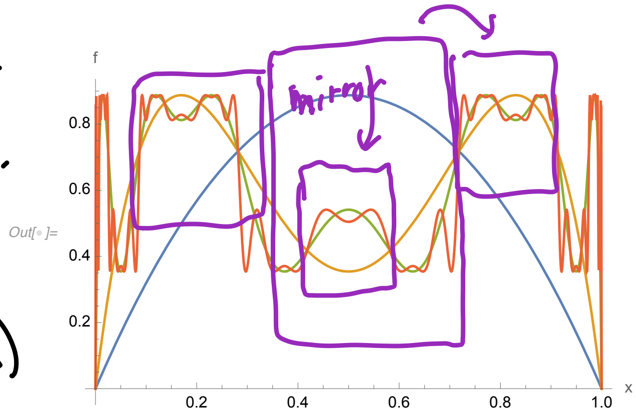
Attractor: set of points x_n $(n \to \infty)$ starting from generic x_0



Recall: iterated map $f^{[n]}(x) = f(f(\cdots f(x)))$

[apply f n times]

Look at $r = 3.55$. (just before chaos)



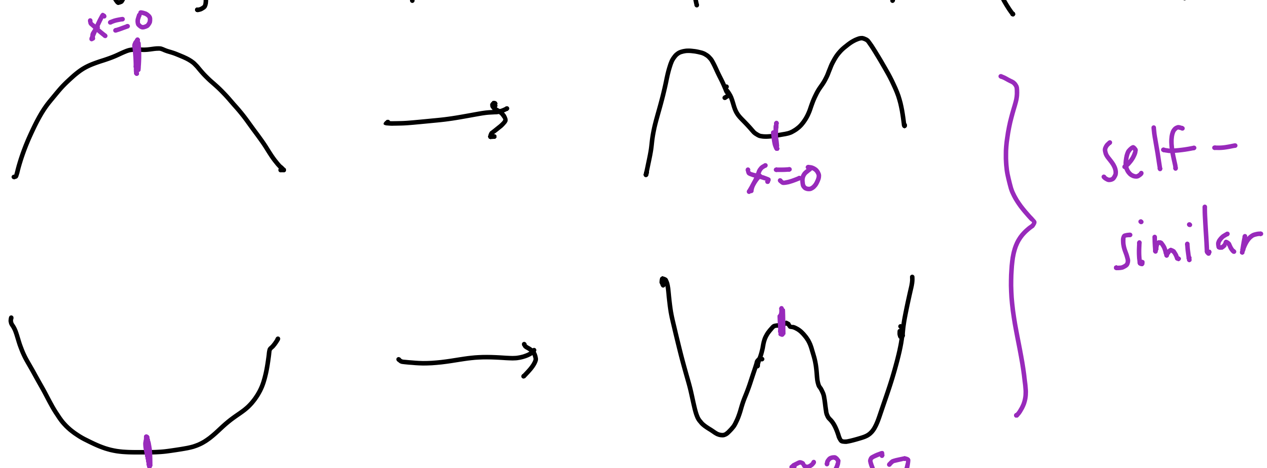
$\to f^{[1]}$ (2nd order poly)

$\to f^{[2]}$ (4th order)

$\to f^{[4]}$ (16th order)

$\to f^{[8]}$ (256th order)

Notice: going from $f^{[2^k]} \longrightarrow f^{[2^{k+1}]} = f^{[2^k]}(f^{[2^k]}(x))$

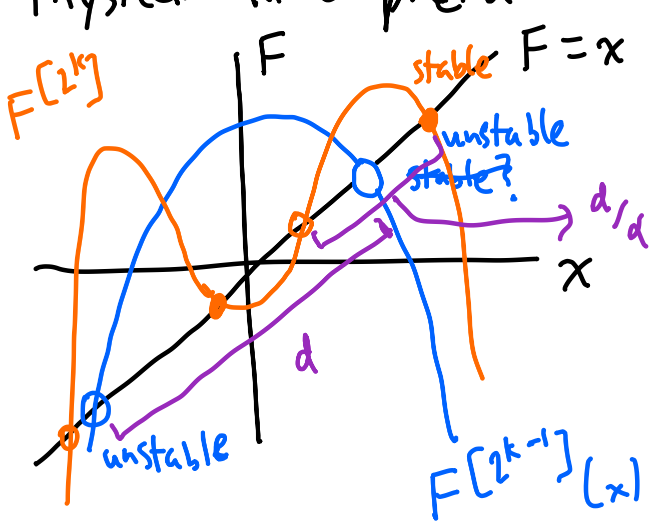


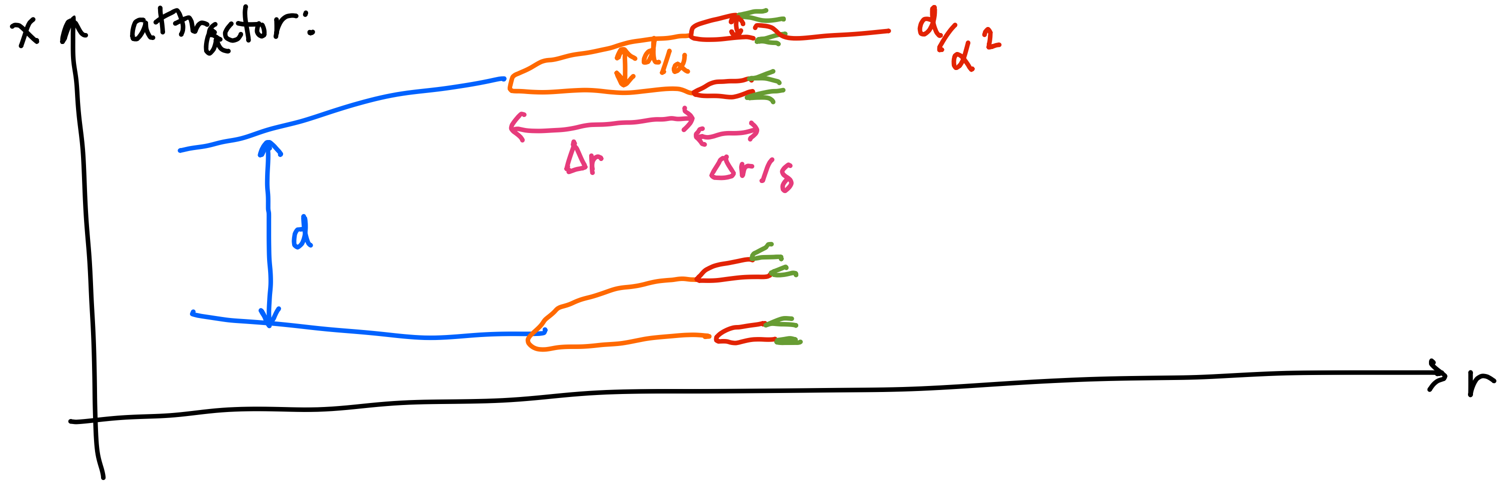
Right at onset of chaos [$r = r_c$ (≈ 3.57)], postulate for shifted function $F^{[2^k]}(x)$:

$$F^{[2^k]}(x) \approx -\frac{1}{\alpha} F^{[2^{k-1}]}(-\alpha x)$$

for some $\alpha > 1$, near $x = 0$.

Physical interpretation:





Feigenbaum numbers α & δ: characterize self-similar behavior of attractor for $r \approx r_c$.

They're universal [indep. of $f(x) = rx(1-x)$], only depend onset of chaos due to period doubling.

<u>Conjecture</u>: (renormalization group): rescaled & iterated maps tend to universal function $g(x)$ — RG fixed point (at $r = r_c$):

$$\lim_{k\to\infty} (-\alpha)^k F^{[2^k]}\left(\frac{x}{(-\alpha)^k}\right) \longrightarrow g(x),$$

independent of original $F^{[1]} = rx(1-x)$.

Use idea to estimate α & $g(x)$?

$$F^{[2^{k+1}]}(x) \approx -\frac{1}{\alpha} F^{[2^k]}(-\alpha x)$$

$$\hookrightarrow F^{[2^k]}(x) = -\alpha F^{[2^{k+1}]}\left(-\frac{x}{\alpha}\right) = -\alpha F^{[2^k]}\left(F^{[2^k]}\left(-\frac{x}{\alpha}\right)\right)$$

$$\frac{1}{\cancel{(-\alpha)^k}} g\left((-\alpha)^k x\right) = \frac{-\alpha}{\cancel{(-\alpha)^k}} g\left(\cancel{(-\alpha)^k} \cdot \frac{1}{\cancel{(-\alpha)^k}} g\left(-\frac{1}{\alpha}\cdot(-\alpha)^k x\right)\right)$$

Set $z = (-\alpha)^k x \longrightarrow$ $\boxed{g(z) = -\alpha\, g\left(g\left(-\frac{z}{\alpha}\right)\right)}$

Solve this equation for $g(z)$ and α.

Approximately!

Note: if $g(x)$ solves equation, so will $\lambda g\left(\frac{x}{\lambda}\right)$.

$$\lambda g\left(\frac{x}{\lambda}\right) = -\alpha \lambda g\left(\frac{1}{\lambda} \cdot \lambda g\left(-\frac{1}{\alpha} \cdot \frac{x}{\lambda}\right)\right)$$

$$\text{or} \quad g(x/\lambda) = -\alpha g\left(g\left(-\frac{x}{\alpha\lambda}\right)\right) \qquad \checkmark$$

So $g(z) = g_0 + g_2 z^2 + \cancel{g_3 z^3 + \cdots}$

↳ fix $g_0 = 1$

Approximate: $g(x) = 1 + g_2 x^2$.

Solve α, g_2 self-consistently?

$$g(x) \approx -\alpha g\left(g\left(-\frac{x}{\alpha}\right)\right)$$

$$1 + g_2 x^2 \approx -\alpha\left[1 + g_2\left(1 + g_2\left(-\frac{x}{\alpha}\right)^2\right)^2\right]$$

$$\approx -\alpha(1+g_2) - \alpha g_2 \cdot 2\frac{g_2}{\alpha^2} x^2 + \cdots$$

$1 = -\alpha(1+g_2)$ $\qquad$ $g_2 = -\frac{2g_2^2}{\alpha}$ or $g_2 = -\frac{\alpha}{2}$

So: $0 = 1 + \alpha + \alpha g_2 = 1 + \alpha - \frac{\alpha^2}{2}$.

Solve: $\alpha = -\left(-1 \pm \sqrt{1+2}\right)$.

Physical solution: $\alpha = 1 + \sqrt{3} \approx 2.73$

Keep higher order terms: $\alpha \approx 2.502\ldots$