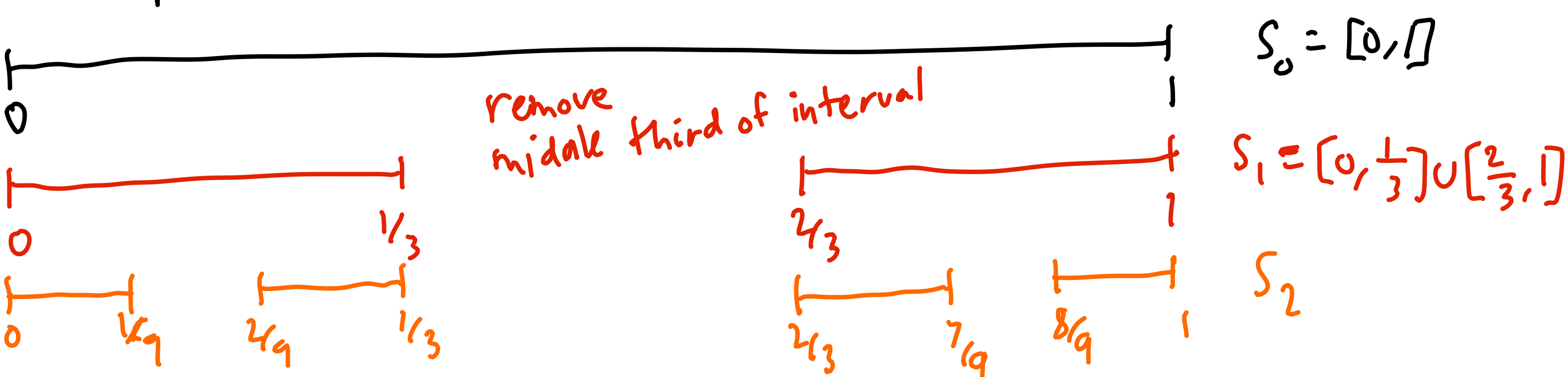
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 39
Fractals

December 4

Fractal = set contain "structure" at arbitrarily small scale

Example 1: Cantor set.



Cantor set "$S = \lim_{n\to\infty} S_n$".

S is not empty ... uncountable! $0, 1/9, 1/27 \ldots \in S$.

What is volume? $[\mathrm{Vol}(S)]$.

$$\mathrm{Vol}(S_0) = \int_0^1 dx = 1$$

$$\mathrm{Vol}(S_1) = 2 \cdot 1/3 = 2/3 = 2/3\, \mathrm{Vol}(S_0)$$

$$\mathrm{Vol}(S_2) = 2/3\, \mathrm{Vol}(S_1) = (2/3)^2 = 4/9$$

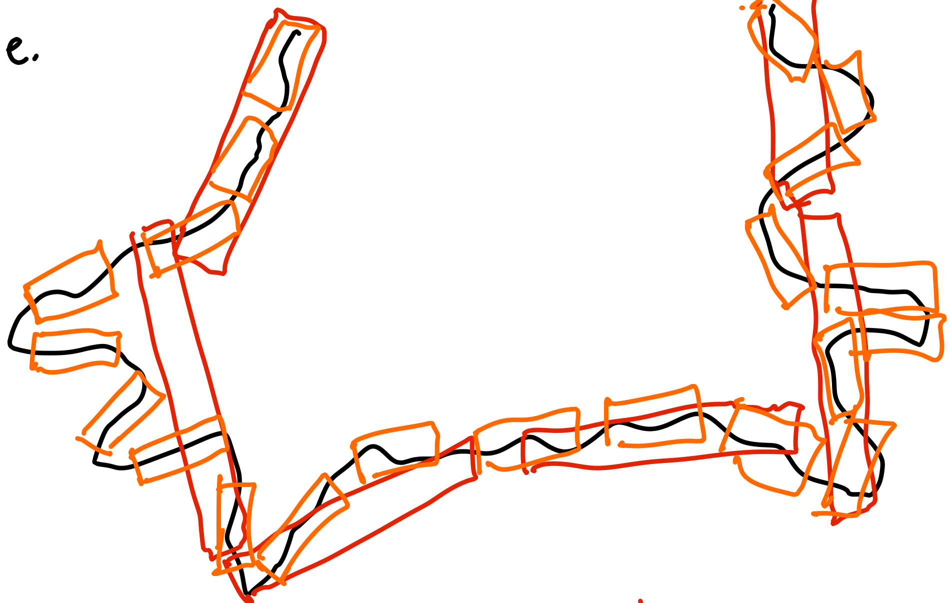
$$\mathrm{Vol}(S) = (2/3)^\infty = 0.$$

Cantor set is uncountable, but "no volume"

Example 2: coastline.

$L = 100$ km

$L = 40$ km



numbers disagree! $\left\{ \begin{array}{l} \text{So} \quad L_{tot} \approx 600 \text{ km} \\ \quad\quad L_{tot} \approx 840 \text{ km} \end{array} \right.$

Claim: disagreement "fundamental". Not clear "smallest scale". Coastline is fractal.

with small enough $L_{ruler} \to 0$, $L_{tot} \to \infty$.

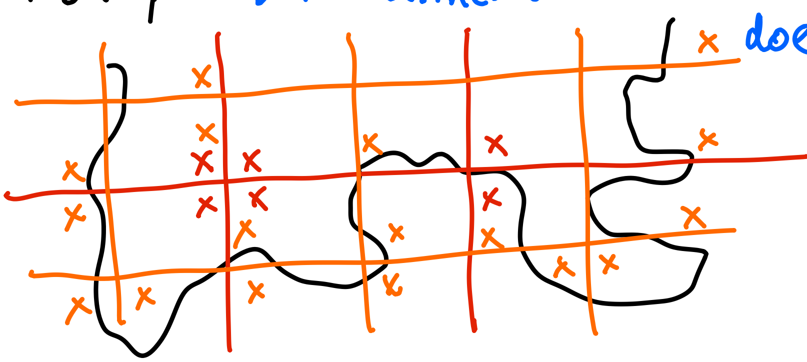
Better idea: length of coastline depends on L_{ruler}:

$$L_{tot}(L_{ruler}) \qquad L_{tot} \sim \frac{1}{L_{ruler}^{d}} \qquad (d \approx 1.2)$$

non-integer dimension d (≈ 1.2) characterizes fractal geometry.

Many notions of dimension for fractal.

Today: box dimension: "how many boxes of scale L_{ruler} does fractal fill?"



Scale L_0

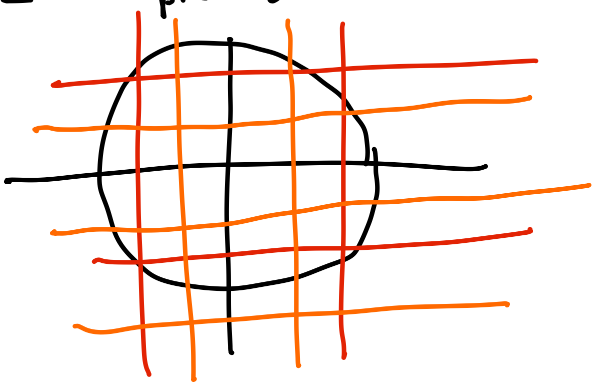
Scale $L_1 = \frac{1}{2} L_0$

let N_n = # of boxes curve passes through at scale L_n.

$N_0 = 6$ $\qquad N_1 = 17$

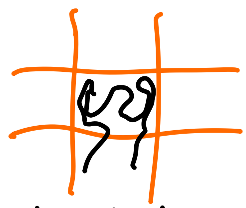
The box dimension: $d = \lim\limits_{\substack{n\to\infty \\ (L_n \to 0)}} \dfrac{\log(N_n)}{\log(1/L_n)}$. i.e. $N_n \sim \dfrac{1}{L_n^d}$

Example 3: circle (embed in $\mathbb{R}^2$)

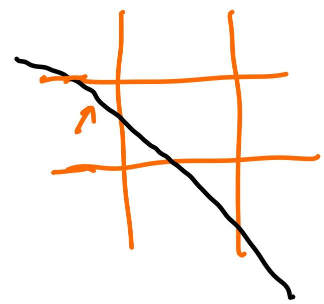


Estimate N_n?

$N_n \approx \dfrac{2\pi R}{L_n} \lesssim 2 \cdot \dfrac{2\pi R}{L_n}$



doesn't happen!



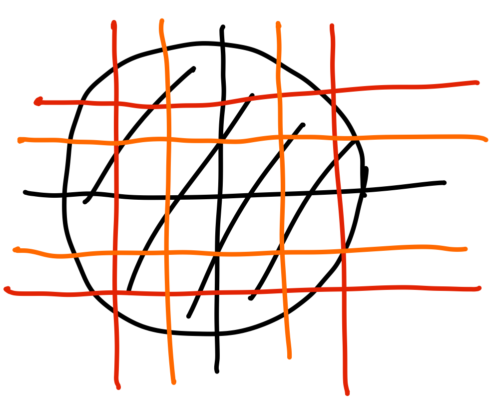
($c = O(1)$)

Box dimension: $d = \lim\limits_{L_n \to 0} \dfrac{\log\left(\frac{2\pi R}{L_n} \cdot c\right)}{\log(1/L_n)}$

$= \lim\limits_{L_n \to 0} \dfrac{\log(1/L_n) + \log(2\pi R \cdot c)}{\log 1/L_n} = 1$

circle is 1-dimensional shape !! ☺

Example 4: disk (fill in circle)



$N_n \approx \dfrac{\pi R^2}{L_n^2}$ (error vanishes as $L_n \to 0$)

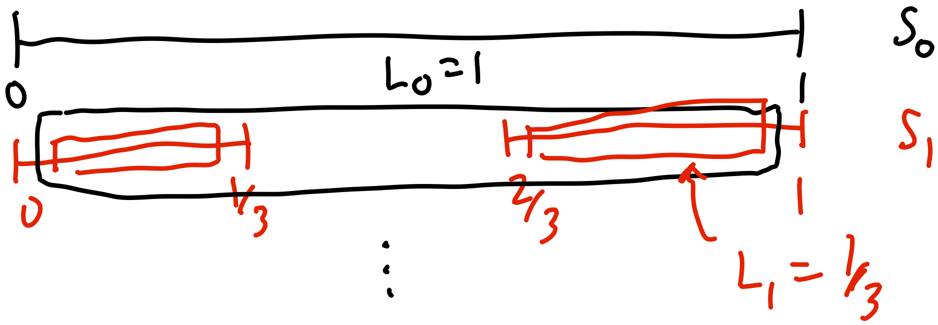
Box dimension:

$d = \lim\limits_{n\to\infty} \dfrac{\log(\pi R^2 / L_n^2)}{\log(1/L_n)}$

$= \lim \dfrac{2\log(1/L_n) + \log(\pi R^2)}{\log(1/L_n)} = 2.$

Box dimension correctly characterizes ordinary shapes with integer d, regardless of "embedding dimension".

Example 1 (again): Cantor set.

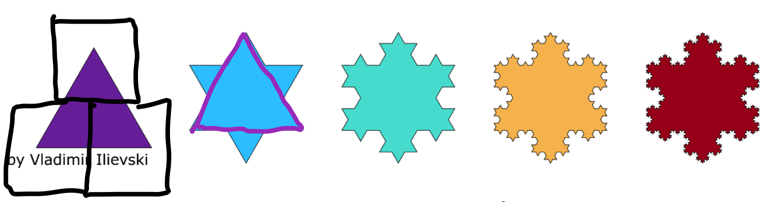


Use $L_n = 3^{-n}$. Cantor set S can be covered using $N_n = 2^n$, since $S \subset S_n$, S_n covered by 2^n boxes.

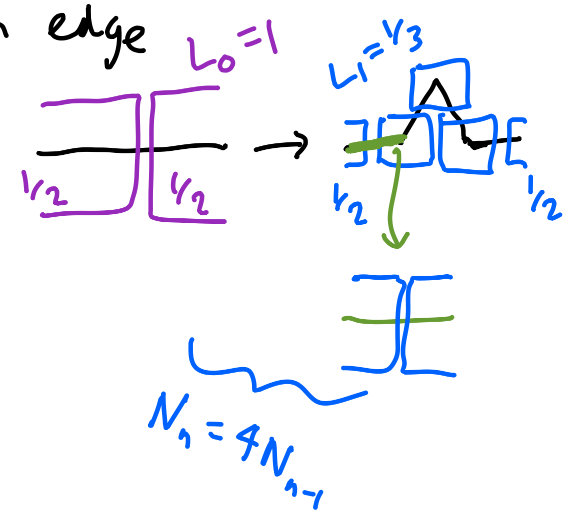
Box dimension: $$d = \lim_{n\to\infty} \frac{\log(N_n)}{\log(1/L_n)} = \lim_{n\to\infty} \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3} \approx \boxed{0.63}$$

Fractal has non-integer dim.

Example 5: Koch snowflake.



each edge



Box side length: $L_n = 3^{-n}$.

And $N_n = \dfrac{3 \cdot P_n}{L_n}$ ← perimeter of iteration n

$= 3 \cdot 4^{n-1}$

Box dimension:
$$d = \lim_{n\to\infty} \frac{\log(\frac{3}{4} \cdot 4^n)}{\log(3^n)} = \lim_{n\to\infty} \frac{n \log 4 + \log \frac{3}{4}}{n \log 3} = \frac{\log 4}{\log 3} \approx 1.26$$