PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 4
Time-translation and boost symmetries

September 6

Noether's Thm: continuous symmetry

$$t \to \tilde{t} = t + \varepsilon T \qquad (\varepsilon \text{ infinitesimal})$$

$$x_i \to \tilde{x}_i = x_i + \varepsilon X_i(t, x_j)$$

$$= x_i + \varepsilon \left(X_i(\tilde{t}, x_j) - \dot{x}_i(\tilde{t}, x_j)\, T \right)$$

and $L(\tilde{t}, \tilde{x}_i) = L(t, x_i) + \frac{d}{dt}(\varepsilon \Phi)$

$\hookrightarrow$ conserved quantity

$$Q = TL + \underbrace{\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{x}_i}(X_i - T\dot{x}_i)}_{} + \Phi$$

write as $\frac{\partial L}{\partial \dot{x}_i}(X_i - \dot{x}_i T)$

(summation convention)

Example 4: time-translation symmetry:

$L(x_i, \dot{x}_i)$ independent of t, so $X_i = 0$

$$t \to t + \varepsilon \quad \begin{cases} T = 1 \\ \Phi = 0 \end{cases}$$

$$Q = L - \sum_{i=1}^{n} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = -E \quad \text{(energy)}$$

If e.g. non-relativistic particle:

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - V = (m\dot{x}) \cdot \dot{x} - L = \frac{1}{2} m \dot{x}^2 + V(x)$$

Is E (defined above) always energy (as usually defined?)

No! → HW 2. (rotating ref frames)

Example 5: Galilean boosts.

Non-relativistic boost: $\left.\begin{array}{l} x(t) \to x(t) - ut \\ t \to t. \end{array}\right\} \begin{array}{l} X = -ut \\ \underline{T = 0} \end{array}$

Invariant building blocks:

if L depends on t

$$0 = \frac{\partial L}{\partial x} X + \frac{\partial L}{\partial \dot{x}} (\dot{X} - \cancel{\dot{T} \dot{x}}) + \cancel{\dot{T} L} + \dot{\Phi} \cancel{\left(+ \frac{\partial L}{\partial t} T \right)}$$

boost

$$0 = -\cancel{u} t \frac{\partial L}{\partial x} - \cancel{u} \frac{\partial L}{\partial \dot{x}} + \left(\dot{\Phi} / u \right)$$

since u is const.

First attempt: $\Phi = 0$.

$$\frac{\partial L}{\partial \dot{x}} + t\frac{\partial L}{\partial x} = 0$$

$\left[\frac{d\dot{x}}{ds} = 1, \quad \frac{dx}{ds} = t \quad \dots\right]$

$\frac{dx}{d\dot{x}} = t$ inv BB

$\downarrow$

$x = t\dot{x} + C$

Solve by $\underline{L = f(x - \dot{x}t, t)}$

Boost invariant.

But have neither x-trans or t-trans symmetry...

[multiple symmetries / don't commute]

Aside: x & t-trans symmetry: $\underline{L = g(\dot{x})}$ incompatible!

Option 1: need $\ddot{x}$.

$\hookrightarrow$ is invariant under boost, t & x-trans

$$L = h(\ddot{x}).$$

Option 2: $\Phi \neq 0$ [L not invariant, S is]

1) x-trans: $\frac{\partial L}{\partial x} = 0$

2) t-trans: $\frac{\partial L}{\partial t} = 0$

$\Big\}$ $\underline{L = g(\dot{x})}$

3) boost: $\frac{\partial L}{\partial \dot{x}} + t\underbrace{\cancel{\frac{\partial L}{\partial x}}}_{=0} + \dot{\Phi} = 0$

$$\boxed{a + b\dot{x}} + \cancel{c\dot{x}^2} + \dots$$

$\frac{d}{dt}(at)$ $\quad$ $\frac{d}{dt}\underbrace{(bx)}_{\Phi}$

NOT total d's

$\int dt\, \dot{x}^2$ = not local...

So: $L = g(\dot{x}) = \underbrace{\cancel{g_0}}_{\text{const.}} + \underbrace{\cancel{a\dot{x}}}_{\text{total d.}} + \frac{1}{2}b\dot{x}^2$

const. doesn't change EOM

re-label $b \to$ mass m

$$L = \frac{1}{2}m\dot{x}^2$$

unique L w/ trans & boost.

Claim: most general effective theory is

$$L = \tfrac{1}{2}m\dot{x}^2 + \underbrace{h(\ddot{x})}_{\text{ignore! why?}} \longrightarrow S = \int dt\, L$$

x&t trans, boost inv.

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}c\,\ddot{x}^2 + \cdots$$

Neglect c-term if $c\ddot{x}^2 \ll m\dot{x}^2$, or $\underbrace{\dfrac{\ddot{x}}{\dot{x}} \ll \sqrt{\dfrac{m}{c}}}$

heuristic:

$$\frac{\cancel{\Delta x}/\Delta t^{\cancel{2}}}{\cancel{\Delta x}/\cancel{\Delta t}} \ll \sqrt{\frac{m}{c}}$$

$$= 1/\Delta t$$

microscopic/ non-universal

or $\Delta t \gg \sqrt{\dfrac{c}{m}} = \tau$

(long time scales)

If interested in $t \gg \tau$,

$$L = \frac{m}{2}\left[\dot{x}^2 + \cancel{\tau^2\ddot{x}^2} + \cdots\right]$$

Usually, effective theory means neglecting higher derivatives, physics on short t & x.

Noether Theorem for boosts: $X = -t$, $T = 0$, $\Phi = mx$

conserved: $Q = -t\dfrac{\partial L}{\partial \dot{x}} + \Phi = m(x - \dot{x}t)$

Also, Noether for x-trans: $p = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} = \text{conserved}$

so $\underline{Q} = mx - pt$ is conserved.

rewrite: mx_0 ← const.

$x = x_0 + \left(\dfrac{p}{m}\right)t$: fixes soln to equation of motion!