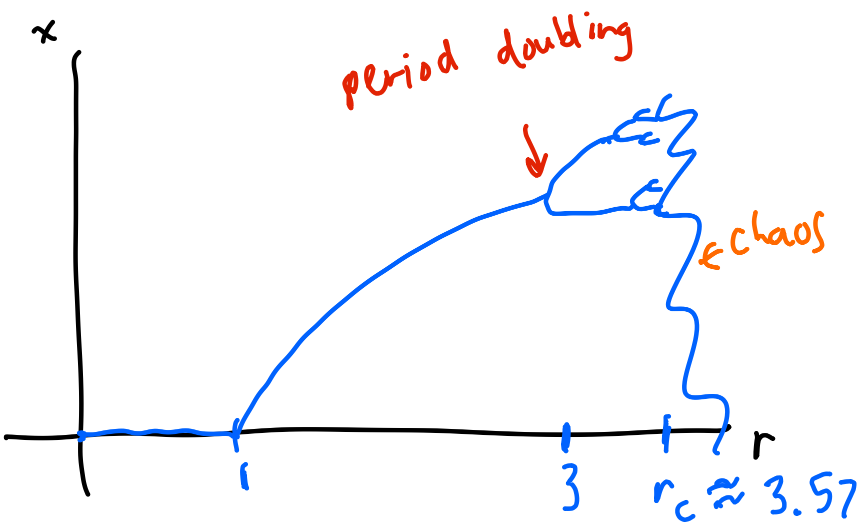
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 40
Fractal structure of attractors

December 6

Logistic map: $x_{n+1} = r\,x_n(1-x_n)$ $\qquad 0 \le x_n \le 1$, $\quad 0 \le r \le 4$

Attractor: (for fixed r), set of points approached by x_n as $n \to \infty$, start from generic x_0.



Claim:

At $r = r_c$, attractor of logistic map is fractal, box dimension $d \approx 0.54$

$$d = \lim_{L_n \to 0} \frac{\log(N_n)}{\log(1/L_n)} \qquad \left[N_n \sim \frac{1}{L_n^{\,d}} \right]$$

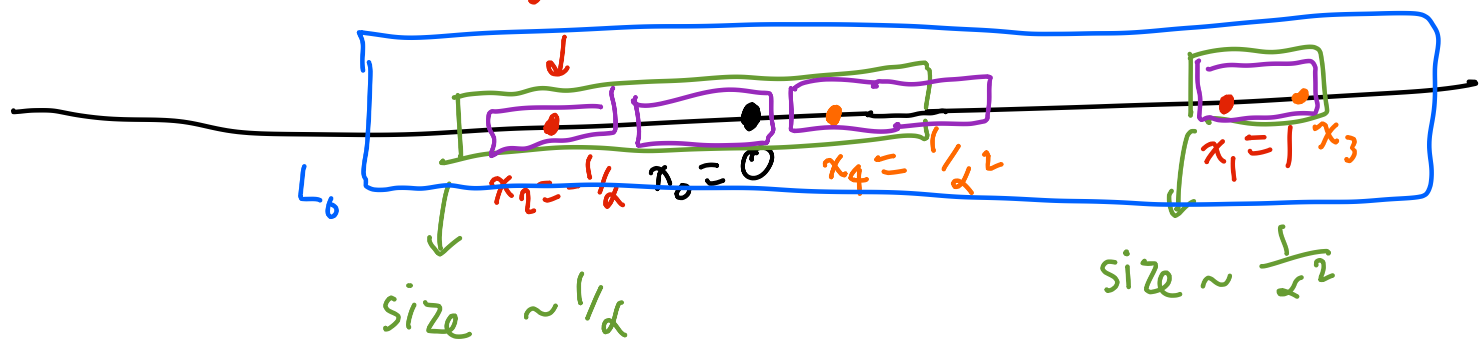
Fractal structure of attractor:

Lec 38: universal function (RG fixed point)

$$g(x) = -\alpha\, g\left(g\left(-\frac{x}{\alpha}\right)\right) \qquad -\frac{1}{\alpha}\, g(-\alpha x) = g(g(x)).$$

Iterate this map many times... Start at $x_0 = 0$.

Choosing $g(0) = 1$: $g(g(0)) = g(1) = -\tfrac{1}{\alpha}\, g(0)$



Estimate the box dimension:

$$\left.\begin{array}{l} \text{If } L_0 \sim 1 \\ \text{then } L_1 \sim 1/\alpha^2 \end{array}\right\} \quad L_n = \frac{L_0}{\alpha^{2n}}$$

[Recall $\alpha \simeq 2.5$]

Count how many boxes N_n?

Strictly $N_1 = 4$ ... but "morally" $N_1 = \alpha + 1$

$$\downarrow$$

$$N_n \approx (\alpha+1)^n$$

Box dimension:

$$d \simeq \lim_{n\to\infty} \frac{\log N_n}{\log\left(\frac{1}{L_n}\right)} = \lim_{n\to\infty} \frac{\log\left((\alpha+1)^n\right)}{\log\left(\frac{\alpha^{2n}}{L_0}\right)}$$

$$= \frac{\log(\alpha+1)}{2\log\alpha} \simeq 0.68 > 0.54.$$

(structure of actual attractor a bit more complex)

Note: structure crudely similar to Cantor set.

Example 2: Lorenz equations.

$$\dot{x} = \sigma(x - z)$$
$$\dot{y} = \underline{x}(r - \underline{z}) - y$$
$$\dot{z} = \underline{xy} - bz$$

σ, r, b dimensionless

only 2 nonlinear terms

Parameter regimes exist where there's chaos:

Poincáre-Bendixson: t-ind ODES, if dissipative,
need 3 DOF for chaos.

Dynamics as $t \to \infty$ tends towards "strange attractor"
(fractal)

Plot of dynamics at: $\sigma = 10$, $r = 28$, $b = 2.67$

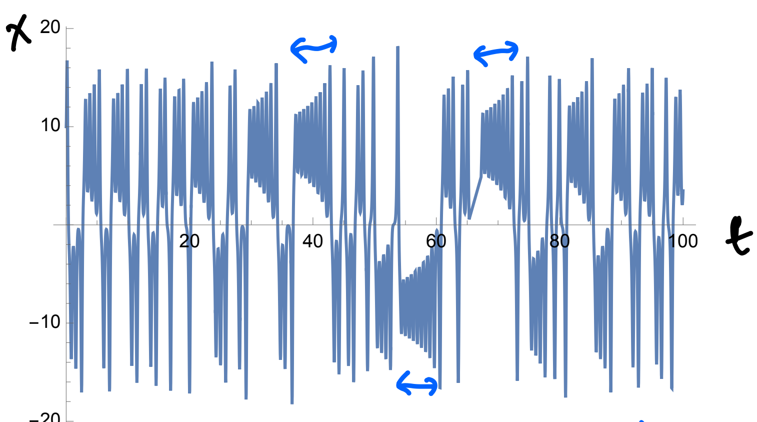
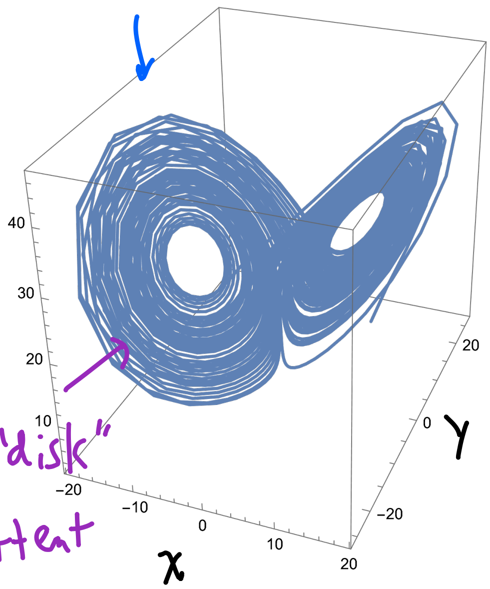


Exhibit: intermittency (HW13): long periods of time
w/ "predictable behavior"

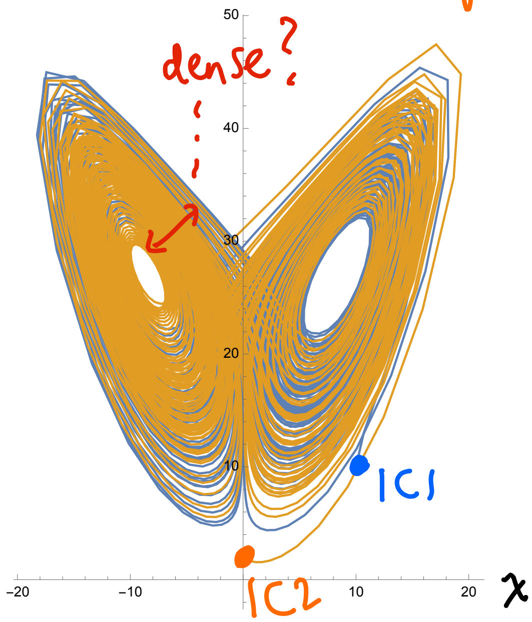
↓

leads to fractal strange attractor. (independent of initial conditions) parametric plot for one trajectory

different ICs → same strange attractor.



each "disk" → intermittent



Numerically ⤳ box dimension for strange attractor is $\underline{d \approx 2.05}$.

disk have "thickness"