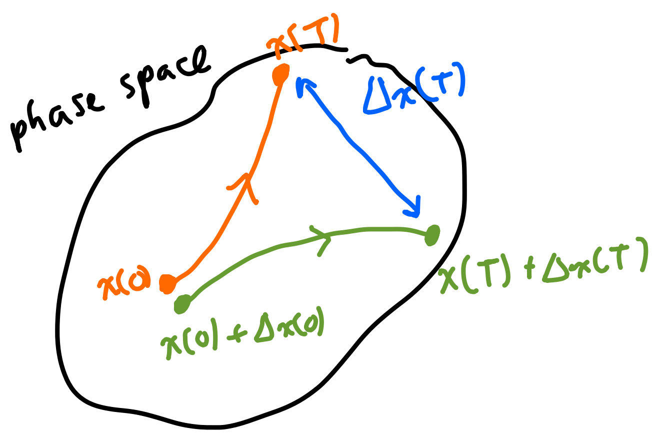
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 41
The butterfly effect

December 8

Butterfly effect: exponential sensitivity of chaotic dynamics to initial conditions.



Lyapunov exponent

$$\Delta x(T) \sim e^{\lambda T}$$

Can write as:

$$\left|\frac{\partial x(T)}{\partial x(0)}\right| \sim e^{\lambda T}$$

Is butterfly effect good diagnostic for chaos?

Δ No? exponential sensitivity near saddle point in Hamiltonian mechanics:

(HW9) $H = xp$ $\qquad \{x,p\}=1$

$\hookrightarrow \dot{x} = \frac{\partial H}{\partial p} = x \qquad$ so $\qquad x(T) = x(0)e^{\lambda T} \qquad (\lambda = 1)$

But $p(T) = p(0)\, e^{-T}$ (Liouville Thm).

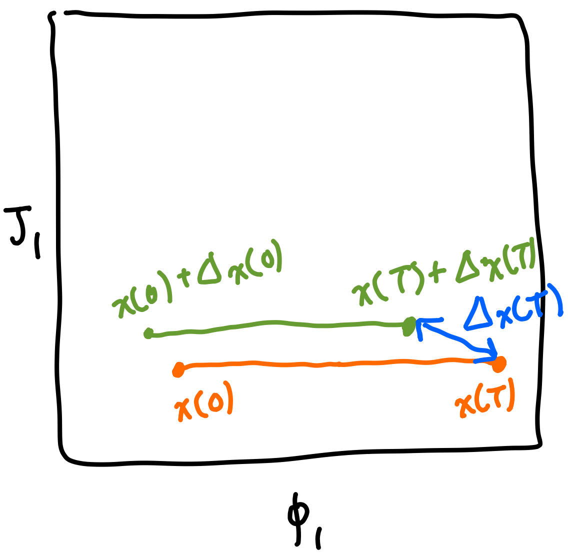
Still true that for "generic" $z = ax + bp$, $\left|\frac{\partial z(T)}{\partial z(0)}\right| \sim e^{T}$ $(T \to \infty)$

D YES? (after addressing caveat above)

In Hamiltonian mechanics, integrability → action-angle variables?

↳ $H(J_A)$:

$$\dot{\phi}_A = \frac{\partial H}{\partial J_A} = \text{const.} \qquad \dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0.$$



$$\Delta x(T) \sim \sqrt{\Delta J_1(T)^2 + \Delta \phi_1(T)^2}$$

$$\sim \sqrt{\underbrace{\Delta J_1^2}_{\dot{J}=0} + (\Delta \phi_1(0) + T \cdot \Delta \omega_1)^2}$$

$\Delta\omega_1 = \frac{\partial H}{\partial J_1} \sim \Delta J$

$$\sim T \cdot \Delta J$$

↳ vs. $e^{\lambda T}$ for butterfly...

See no butterfly effect in an integrable system for generic initial conditions.

lec 42: chaos via proliferation of saddles?

Example: logistic map: $x_{n+1} = r x_n (1 - x_n)$ $\quad 0 \le x_n \le 1$, $0 \le r \le 4$

choose $r = 3.6$ so chaos

Study trajectories:

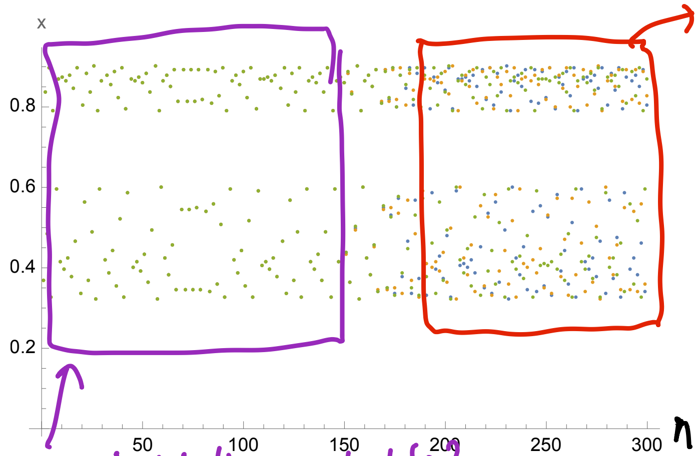
$x_0, x_1, x_2, \ldots$

$x_0 + \delta x_0,\ x_1 + \delta x_1,\ x_2 + \delta x_2$

For generic ICs, $\delta x_n \to \mathcal{O}(1)$ as $n \to \infty$? How fast?

Numerical experiment! 3 runs w/ similar init. cond:

$x_0 = 0.37$ $\qquad$ $x_0 = 0.37 + \underbrace{10^{-15}}$ $\qquad$ $x_0 = 0.37 - 10^{-15}$



all 3 trajectories appear different

↓

chaos!

butterfly effect.

trajectories indistinguishable?

Estimate Lyapunov exponent?

$$\underbrace{\delta x_n}_{1} \sim \underbrace{\delta x_0}_{10^{-15}} \cdot e^{\lambda n}$$

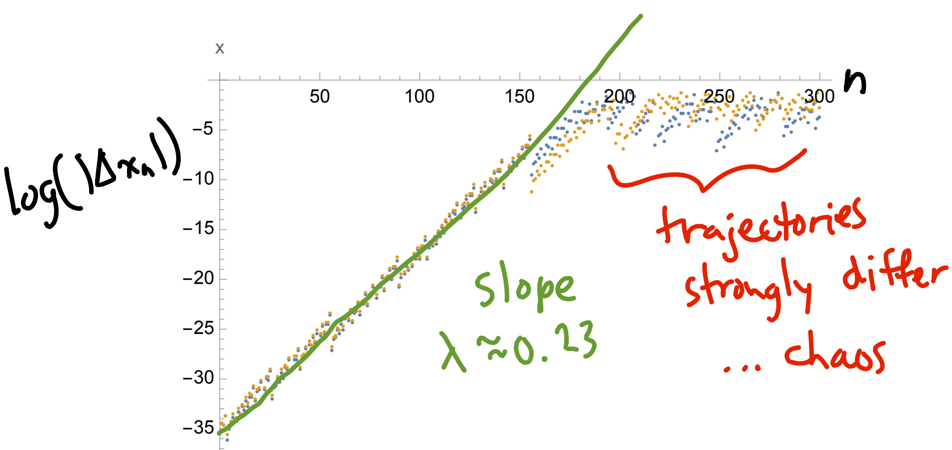
$$1 \quad 10^{-15} \cdot e^{\lambda \cdot 150}$$

$$10^{15} \sim e^{150\lambda}$$

$$15 \log(10) \approx 150\lambda \rightarrow \lambda \approx 0.23$$

Note: λ independent of x_0 (more later)

Confirm exponential growth directly.

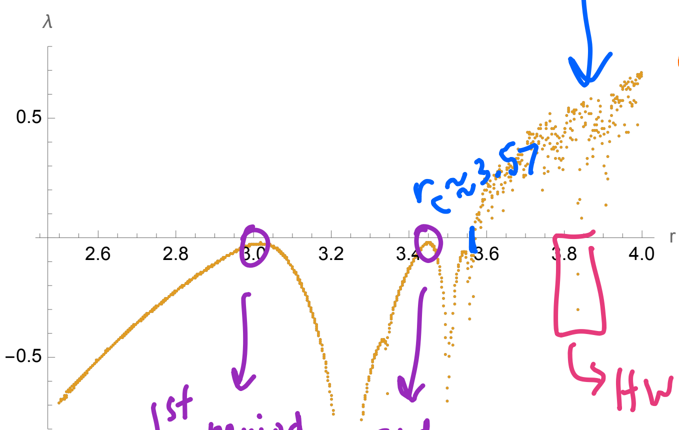


Exponential growth crucial to butterfly effect.

$$\Delta x_n \sim \Delta x_0 e^{\lambda n}$$

$$\log|\Delta x_n| \approx \log|\Delta x_0| + \lambda n$$

Claim: $\lambda > 0$ heralds chaos in logistic map



Calculated λ from $x_0^{(1)}$ & $x_0^{(2)}$.
(Measure slope of above line...)

Repeat at multiple r...
↳ λ independent of x_0.

1st period doubling

2nd period doubling

↳ HW13: stable period-3 cycle

Associated w/ $\lambda \to 0$ (from below!) because e.g. approach $r \to 3^-$, one stable fixed pt becoming unstable. (lec 37) linear stability...

$x_* + \delta x_n \sim \lambda_* (x_* + \delta x_{n-1})$ $\qquad \lambda_* \sim e^{\lambda}$

fix pt

↳ $|\lambda_*| \to 1$ as $r \to 3$.

↳ $\lambda \to 0$