PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 42
KAM Theorem

December 11

Recall: integrable Hamiltonian has action-angle variables (ϕ_A, J_A)

$\downarrow H_0(J_A)$

$$\dot{J}_A = -\frac{\partial H_0}{\partial \phi_A} = 0 \qquad \text{and} \qquad \dot{\phi}_A = \frac{\partial H_0}{\partial J_A} = \omega_A = \text{const.}$$

In perturbation theory (lec 33)

$$H = H_0(J_A) + \varepsilon H_1(\vec{\phi}, \vec{J}) \rightarrow H_1 = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} e^{i\vec{m}\cdot\vec{\phi}} h_{\vec{m}}$$

Goal: find new action-angle variables via Type 2 CT:

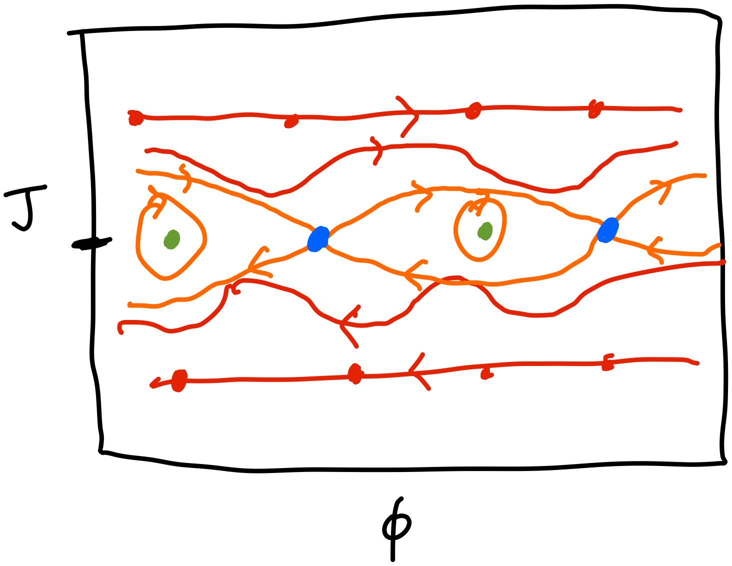
generated by $S_1(\vec{\phi}_0, \vec{J}) = \sum_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}} \frac{h_{\vec{m}}(\vec{J})}{\underline{i\vec{m}\cdot\vec{\omega}}}$

Perturbation theory sick if: $\vec{m}\cdot\vec{\omega} = 0$ (commensurate freq.)

even if $\varepsilon \to 0$, PT won't work

J's where such breakdowns occur can be dense...

AA vars "break" first near points in phase w/ commensurate frequencies...



Recall Lec. 33 and HW12

Resonance at "$n=2$":

We have 2 stable points

and 2 unstable points

and always equal # of each.

"Stable" points $\longrightarrow$ "islands of integrability"

"Unstable" points $\longrightarrow$ chaos

Consider discrete map (e.g. Kicked rotor):

$$\begin{pmatrix} J_{n+1} \\ \phi_{n+1} \end{pmatrix} = \begin{pmatrix} J_n + \varepsilon \sin\phi_n \\ \phi_n + J_n + \varepsilon \sin\phi_n \end{pmatrix} = Z \cdot \begin{pmatrix} J_n \\ \phi_n \end{pmatrix}$$

NOT matrix

IMPORTANT: Z^{-1} exists b/c can run Hamilton's eqs backwards.

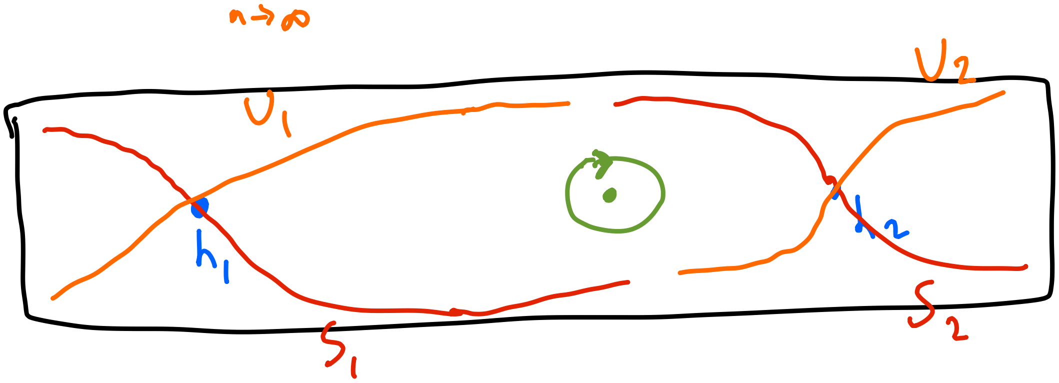
For "unstable" (hyperbolic) fixed point h_i: $[Z \cdot h_i = h_i]$

S_i = "stable manifold", set of points where

$$\lim_{n\to\infty} Z^n \cdot x = h_i \quad \text{if} \quad x \in S_i.$$

U_i = "unstable manifold", set of points where

$$\lim_{n\to\infty} Z^{-n} \cdot x = h_i, \quad \text{if} \quad x \in U_i$$



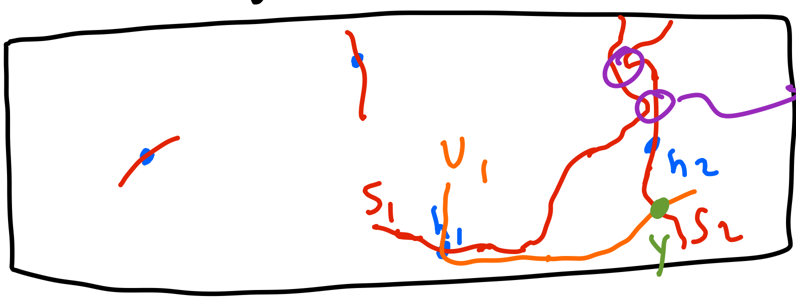
Claim: S_1, S_2 cannot intersect

Proof: Suppose $x \in S_1$ and $x \in S_2$.

$$\lim_{n\to\infty} Z^n \cdot x = h_1 \qquad \lim_{n\to\infty} Z^n \cdot x = h_2$$

Contradict! so $S_1 \cap S_2 = \phi$ [empty set]

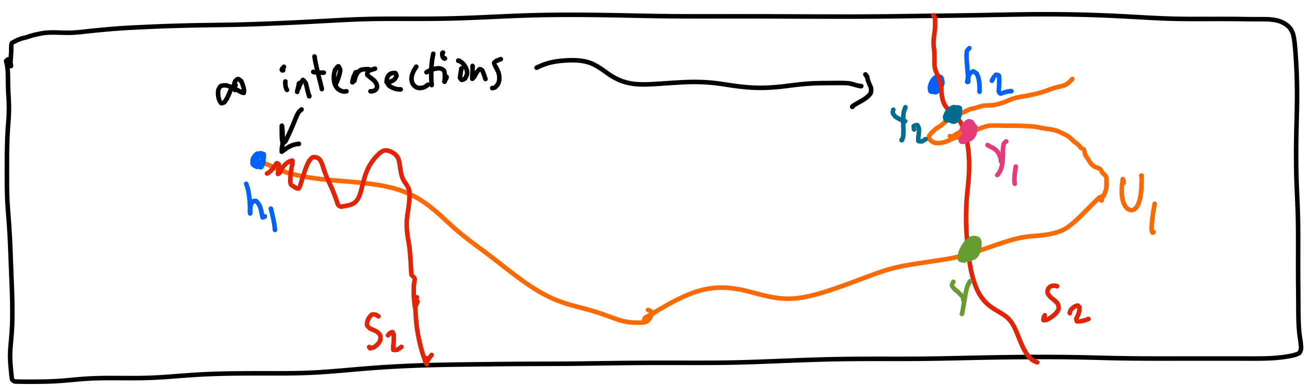
Similar argument: U_1 & U_2 can't intersect.



S_1 & S_2 don't intersect

BUT, S_1 & U_2 CAN intersect. $\qquad y \in S_2 \cap U_1$

Claim: S_1 & U_2 intersect ∞ many times!!



Consider point $y_1 = Z \cdot y$

$$\lim_{n\to\infty} Z^n \cdot y_1 = \lim_{n\to\infty} Z^{n+1} \cdot y = h_2 \quad \& \quad \lim_{n\to\infty} Z^{-n} \cdot y_1 = \lim_{n\to\infty} Z^{-(n-1)} \cdot y = h_1$$

$y_2 = Z^2 \cdot y$, etc...

Intersect at any $Z^m \cdot y$ for any integer m.

Chaos comes from complicated intersections of S_2 & U_1 (homoclinic tangle)

$\Rightarrow$ are we on U_1 vs. U_2 $\rightarrow$ different outcomes...

<u>Claim:</u> chaos exists at any $\varepsilon \neq 0$!
(at least "somewhere" in phase space)

KAM Theorem: If ε small enough, $H = H_0 + \varepsilon H_1$,
dynamics stays <u>integrable</u> in "finite" fraction of phase space (dense)

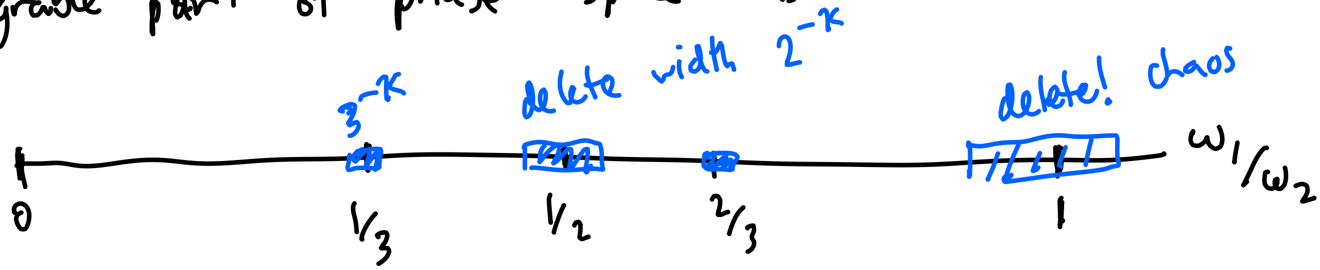
Idea: Perturbation $S_1 \sim \varepsilon \sum_{\vec{m}} \boxed{\frac{h_{\vec{m}}}{i\vec{m}\cdot\vec{\omega}}} e^{i\vec{m}\cdot\vec{\phi}}$ $\rightarrow$ OK is small!

While we can always find $\vec{m}$ $[|\vec{m}| \to \infty]$ w/ $\vec{m}\cdot\vec{\omega} \to 0$,
also have $|h_{\vec{m}}| \sim e^{-|\vec{m}|}$

Diophantine condition: $|\vec{m}\cdot\vec{\omega}| \geq \gamma \cdot |\vec{m}|^{-\varkappa}$ if $\varkappa > n$ (DOF)
for "typical" $\vec{J}$ $(\vec{\omega})$
(incommensurate)

so: $\sum_{\vec{m}} \frac{1}{\gamma} |\vec{m}|^{\varkappa} e^{-|\vec{m}|} \rightarrow$ finite.

Integrable part of phase space has "Cantor-like structure"



Because $\varkappa > n$, only finite fraction of phase space needs to be removed at small ε.
↳ failure of PT to converge