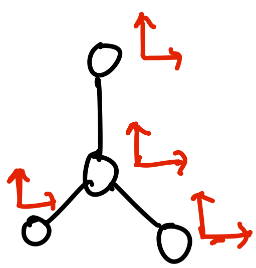
PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 5
Harmonic oscillators

September 8

Goal: universal theory for dynamics near (not unstable) equilibrium.

Example HW2:



$x_i \quad (i=1,\ldots,8)$

Choose coords so $x_i = 0 \to$ equilib.
(implicit: hold for $i=1,\ldots,n$)

Need: $x_i(t)=0$ is valid extremum of $S=\int dt\, L$.

trajectory is const. in t

$$\left.\frac{\delta S}{\delta x_i}\right|_{x_i=0} = 0 \longrightarrow 0 = \frac{\partial L}{\partial x_i} - \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)} + \cdots$$

So $\underbrace{0 = \frac{\partial L}{\partial x_i} \text{ at } x_j = 0}$

$\left(\frac{\partial L}{\partial x_i} = 0 \text{ at } \vec{x} = (0,\ldots,0) \text{ for } i=1,\ldots,n\right)$

Since consts in L unphysical, best thing:

$$L = -\underbrace{\frac{1}{2}K_{ij}\,x_i x_j}_{\sum_{i,j=1}^{n}\ \text{implicit}} + \cdots \quad A_{ijk}x_i x_j x_k + \cdots \quad (\text{or } \dot{x})$$

Take $K_{ij} = K_{ji}$ (K is symmetric):

re-label summed index: $i \to j$, $j \to i$

$$K_{ij}\,x_i x_j = K_{ij}\left(\frac{x_i x_j + x_j x_i}{2}\right) = \frac{K_{ij}\,x_i x_j}{2} + \boxed{\frac{K_{ij}\,x_j x_i}{2}}$$

$$= \left(\frac{K_{ij} + K_{ji}}{2}\right) x_i x_j \quad \text{take } K_{ij} \text{ to be symmetric}$$

Dynamics $\to$ add $\dot{x}$,

Add $V_{ij}\,x_i \dot{x}_j$? $= \frac{d}{dt}\left(\frac{1}{2}V_{ij}x_i x_j\right)$? $\hookrightarrow$ only symmetric V is not needed.

but microscopic time-reversal: $t \to -t$, $\dot{x}_i \to -\dot{x}_i$

Next: $\frac{1}{2}M_{ij}\,\dot{x}_i \dot{x}_j$, $M_{ij} = M_{ji}$

So,
$$L = \underbrace{\frac{1}{2}M_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}K_{ij}\,x_i x_j}_{\text{coupled harmonic oscillator}} + \cdots$$

Let's go ahead, start w/ $n=1$ DOF:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

$$\frac{\delta S}{\delta x} = 0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -kx - m\ddot{x}$$

Set $\omega^2 = k/m$: $\ddot{x} = -\omega^2 x$

integration const.

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Equilibrium $(x=0)$ is not unstable, need ω real, $\frac{k}{m} \geq 0$.

By convention, $m, k \geq 0$.

Corrections to oscillator:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + a \dot{x}^3 + b x^3 + \ldots$$

When can $b \approx 0$?

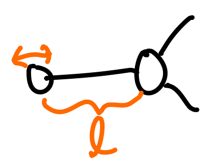
$x(t)$ have small amplitude:

Need $|kx^2| \gg |bx^3|$ or $\underbrace{|x| \ll k/b}$

Write $b = k/\ell$, so $x \ll \ell$.

$$L = -\frac{1}{2} k x^2 + k x^2 \cdot \underbrace{\left(\frac{x}{\ell}\right)}_{\ll 1} + \ldots$$

Physical:

"  "

When $a \to 0$ OK?

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + \underbrace{a \dot{x}^3}$$

$\hookrightarrow a = \frac{m\tau}{\ell}$ so $L = \frac{1}{2} m \dot{x}^2 \left[1 + 2\frac{\tau}{\ell}\dot{x}\right] + \ldots$

$\frac{\dot{x}}{x} \sim \omega$ $\frac{\tau}{\ell}\dot{x} \ll 1$ if $\underbrace{(\omega\tau)}_{\omega\tau \lesssim 1}\left(\frac{x}{\ell}\right) \ll 1$

General n (coupled oscillator):

$$L = \frac{1}{2} M_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} K_{ij} x_i x_j .$$

(implicit $\sum_{i,j=1}^{n}$)

Goal: $L = \sum_{i=1}^{n} L_i$, with $L_i \sim \frac{1}{2}\dot{q}_i^2 - \frac{1}{2}\omega_i^2 q_i^2$ (no sum on i)

① Find eigenvectors of M (diagonalize M):

$x_i = Q_{ij} y_j$ where

$\vec{x} = Q\vec{y}$

$M'_{kl} = M_{ij} Q_{ik} Q_{jl}$ = diagonal

$M' = Q^T M Q$ = diag.

$K'_{kl} = K_{ij} Q_{ik} Q_{jl}$

② Rescale y's so $M' \to$ identity: $\frac{1}{2} M'_{ii} \dot{y}_i^2 \to \frac{1}{2}\dot{z}_i^2$

(no sum on i) $z_i = \sqrt{M'_{ii}}\, y_i$

$K''_{ij} \to \frac{K'_{ij}}{\sqrt{M'_{ii} M'_{jj}}}$.

$$\left[L = \frac{1}{2}\sum_{i=1}^{n} \dot{z}_i^2 - \sum_{i,j} \frac{1}{2} K''_{ij} z_i z_j \right]$$

③ Choose matrix R_{ij} which is orthogonal: $R_{ij} R_{ik} = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j\neq k \end{cases}$

so that $R_{ik} R_{jl} K''_{ij} = W_{kl}$ = diagonal

$z_i = R_{ij} q_j$

$$L = \sum_{i=1}^{n} \left[\frac{1}{2}\dot{q}_i^2 - \frac{1}{2} W_{ii} q_i^2 \right]$$

q_i's = normal modes.

freq. $\sqrt{W_{ii}}$

not unstable $\to$ $W_{ii} \geq 0$.