PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 6
Relativistic particles

September 11

Goal: action for free relativistic particle, mass m

Step 1: identify symmetries

① translation:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} + \underbrace{\begin{pmatrix} \varepsilon_t \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_x \end{pmatrix}}_{4}$$

[coord origin doesn't matter]

② rotation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{3}$$

rotate around x, y, z axis

③ boosts:

$$\begin{pmatrix} ct \\ x \end{pmatrix} \to \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

3 directions to boost in.

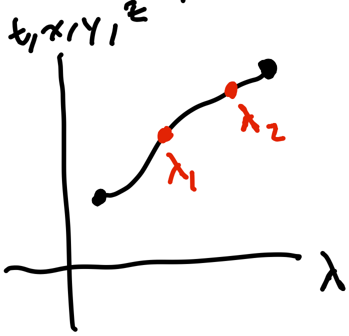
How many continuous symmetries? $4 + 3 + 3 = \underline{10}$.

completely fix S!

Note: t & x on (almost) equal footing.

↳ behave "differently" in $\int dt \, L(\dot{x}, \dot{y}, \dot{z} \cdots)$

Idea: formulate action: $S[t(\lambda), x(\lambda), y(\lambda), z(\lambda)]$.



$\lambda_1 < \lambda_2$ if 1 before 2

ADD: reparameterization symmetry

$\lambda \to \underbrace{f(\lambda)}$, if f mono. increasing

$f_0 + f_1\lambda + f_2\lambda^2 + \cdots$

one cont. sym.

Notation: $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Big\} x^\mu$

Greek indices → space & time

Set $\underline{c=1}$. (work units w/ $c=1$).

Goal: deduce $S[x^\mu(\lambda)] = \int d\lambda \, L\left(x^\mu, \frac{dx^\mu}{d\lambda}, \cdots\right)$

Step 2: Deduce invariant building blocks.

(ignore reparameterizations until end)

① Translations: $x^\mu \to x^\mu + \varepsilon^\mu$ $\begin{pmatrix} \varepsilon_t \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{pmatrix}$

↓ from lec 3

$\frac{\partial L}{\partial x^\mu} = 0.$ (implicit: holds for all μ)

So: $L = L\left(\frac{dx^\mu}{d\lambda}, \cdots\right)$

② & ③: Rotation & boost. = Lorentz.

Rotation: $\underbrace{\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}}$

boost: $\begin{pmatrix}t\\ x\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta\\ -\gamma\beta & \gamma\end{pmatrix}\begin{pmatrix}t\\ x\end{pmatrix}$

z-rotation leaves x^2+y^2 invariant. (length of spatial vector)

x-boost leaves $-t^2+x^2$ invariant. $-(\text{proper time})^2$.

More generally: $-t^2+x^2+y^2+z^2$ invariant under Lorentz

Write as: $\underbrace{\eta_{\mu\nu}x^\mu x^\nu}_{\text{implicit } \sum_{\mu\nu}}$ where $\eta_{\mu\nu} = \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

In relativity: $x_\mu = \eta_{\mu\nu}x^\nu = \begin{pmatrix}-t\\ x\\ y\\ z\end{pmatrix}$ lower index flip sign on x_t.

Lorentz transformation: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ such that $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$

$[\eta = \Lambda^T \eta \Lambda]$

Infinitesimal? $\Lambda^\mu{}_\nu = \underset{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}}{\delta^\mu{}_\nu} + \varepsilon^\mu{}_\nu$?

$$\eta_{\mu\nu} \approx \eta_{\rho\sigma}(\delta^\rho_\mu + \varepsilon^\rho{}_\mu)(\delta^\sigma{}_\nu + \varepsilon^\sigma{}_\nu) + \mathcal{O}(\varepsilon^2)$$

$$\approx \eta_{\mu\nu} + \eta_{\mu\sigma}\varepsilon^\sigma{}_\nu + \eta_{\rho\nu}\varepsilon^\rho{}_\mu$$

or

$$\eta_{\mu\sigma}\varepsilon^\sigma{}_\nu = -\eta_{\rho\nu}\varepsilon^\rho{}_\mu \longrightarrow \varepsilon^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & \beta_x & \beta_y & \beta_z\\ \hline \beta_x & 0 & \theta_z & -\theta_y\\ \beta_y & -\theta_z & 0 & \theta_x\\ \beta_z & \theta_y & -\theta_x & 0 \end{array}\right)$$

boosts

rotations.

To recap: "Spacetime vectors" $a^\mu \to \Lambda^\mu{}_\nu a^\nu$
$b^\mu \to \Lambda^\mu{}_\nu b^\nu$

Invariant: $a^\mu (b^\nu \eta_{\mu\nu}) = a^\mu b_\mu$: have no "free" indices: contract lower & upper index.

all invariants if $t \to -t$ is a symmetry

Hence: minimal invariant BB is: $\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\eta_{\mu\nu} = \frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}$.
(trans.+Lor.)

And $S = \int d\lambda \, L\left(\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}, \dots\right)$

Now invoke reparameterization symmetry: $\lambda \to f(\lambda)$

$$\int d\lambda \to \int d\lambda \, f'(\lambda) = \int d[f(\lambda)]$$
$$\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{df(\lambda)} = \frac{1}{f'(\lambda)}\frac{dx^\mu}{d\lambda}$$

Unique (up to overall constant):

$$S = -m\int d\lambda \sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} = -m\int d\lambda \sqrt{+\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dy}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2}.$$

physical trajectories timelike:

$\Delta t^2 > \Delta x^2 + \Delta y^2 + \Delta z^2$

physical interpretation:

$S = -m\int d\tau$

$= -m \times$ [proper time on worldline]

One parameterization: $\lambda = t$.

$$S = -m\int dt\,\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2} = \int \frac{dt}{\gamma}$$

Restore c:

$$S = -mc^2\int dt\,\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}$$

If nonrelativistic limit: $|\dot{x}| \ll c$, Taylor expand:

$$S = \int dt\left[\cancel{-mc^2} + \frac{m\cancel{c^2}}{2\cancel{c^2}}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \cdots\right]$$

const.
don't matter in L