PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 7
Charged relativistic particles

September 13

Action for free relativistic particle:

$$S = \int \underset{\uparrow}{d\lambda} \left(-m \sqrt{\underbrace{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}} \right) \longrightarrow$$

Choose $\lambda = t$:

$$S = \int dt \left(-m\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \right)$$

reparameterization:
$\lambda \to f(\lambda)$

μ index: summed over $\in \{t, x, y, z\}$

Today: interactions w/ EM field.

What interactions make sense?

$$S[x^\mu(\lambda)] = \cdots + \int d\lambda\, \cancel{f'(\lambda)}\; \underbrace{\left(\cancel{\frac{1}{f'(\lambda)}} \frac{dx^\mu}{d\lambda} \right) K_\mu}$$

Goal: EM fields: $K_\mu = q A_\mu$

$$S[x^\mu(\lambda)] = \cdots + q\int d\lambda \frac{dx^\mu}{d\lambda} A_\mu(x) = \cdots + \int q\, dx^\mu A_\mu$$

Review: (relativistic) E&M:

$$A_\mu \to \begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} -\varphi \\ A_x \\ A_y \\ A_z \end{pmatrix} ; \qquad \vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{3 comp.} \qquad \vec{B} = \nabla \times \vec{A} \quad \text{3 comp.}$$

Relativistically $\underbrace{\text{covariant}}$ form:

transforms "nicely" under Lorentz. (linearly)

$\frac{\partial}{\partial x^\mu} = \partial_\mu$

Guess: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

16 comp. antisymmetric part have $\binom{4}{2} = 6$ ind. comp.

- $F_{tt} = \partial_t A_t - \partial_t A_t = 0$; all diagonal comp of $F_{\mu\nu} = 0$.
- $F_{ti} = -F_{it} = \partial_t A_i - \partial_i A_t = \frac{\partial A_i}{\partial t} - \frac{\partial}{\partial x^i}(-\varphi) = -E_i$

  $i j \cdots =$ spatial $\{x, y, z\}$ only
- $F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk} B^k$

  Levi-Civita: $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$

  $\varepsilon_{xyz} = 1$.

  $\varepsilon_{xxi} = 0 = -\varepsilon_{xxi}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

EM invariant under gauge transformations:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \quad \partial_\mu\Lambda = \frac{\partial \Lambda}{\partial x^\mu}$$

physical fields:

$$F_{\mu\nu} \to \partial_\mu A_\nu - \partial_\nu A_\mu + \cancel{\partial_\mu \partial_\nu \Lambda} - \cancel{\partial_\nu \partial_\mu \Lambda} = F_{\mu\nu}$$

Claimed: $S = q\int \underbrace{d\lambda \frac{dx^\mu}{d\lambda}}_{\text{reparameterization invariant}} A_\mu$

gauge transformations:

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

$$S \to \int d\lambda \frac{dx^\mu}{d\lambda}\, q\,(A_\mu + \partial_\mu \Lambda)$$

$$= q\int dx^\mu A_\mu + \underbrace{q\int dx^\mu \partial_\mu \Lambda}$$

$$= q\left[\Lambda(x^\mu_f) - \Lambda(x^\mu_i)\right]$$

boundary term.

$F^{\rho\sigma} = \eta^{\rho\mu}\eta^{\sigma\nu}F_{\mu\nu}$

$\eta^{\rho\mu} = \eta_{\rho\mu}$

Can we add more invariant building blocks?

Try: $S = \dots + \int d\lambda \frac{dx^\mu}{d\lambda} A_\mu \cdot \underbrace{(F_{\rho\sigma}F^{\rho\sigma})}_{\text{gauge invariant...}}$?

Gauge: $\int d\lambda \frac{dx^\mu}{d\lambda} \underbrace{(\partial_\mu \Lambda) F_{\rho\sigma}F^{\rho\sigma}}$

$$\neq \partial_\mu\left(\Lambda F_{\rho\sigma}F^{\rho\sigma}\right)$$

Thus: reparameterization + Lorentz (covariance) + gauge

"uniquely" fix S.

Pick $\lambda = t$. Euler-Lagrange equations:

$$S = \int dt \Big[-m \sqrt{1 - \underbrace{\dot{x}_j \dot{x}^j}_{\dot{x}^2+\dot{y}^2+\dot{z}^2}} + q\underbrace{(-\varphi)}_{A_t} + q\dot{x}^j A_j \Big]$$

[φ, A_i can depend on t, x^i]

$$\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0.$$

$$\frac{\partial \dot{x}^j}{\partial \dot{x}^i} = \delta^j_i$$

$$0 = \left[-q\partial_i\varphi + q\dot{x}^j \partial_i A_j\right] - \frac{d}{dt}\left[\frac{m\dot{x}_i}{\sqrt{1-\dot{x}_j\dot{x}^j}} + qA_i\right]$$

chain rule

$$\underbrace{\frac{d}{dt}\left[m\frac{\dot{x}_i}{\sqrt{1-\dot{x}_j\dot{x}^j}}\right]}_{\frac{d}{dt}(\text{momentum}) = \text{force}} = q\underbrace{\left[-\partial_i\varphi - \partial_t A_i\right]}_{E_i} + q\dot{x}^j\underbrace{(\partial_i A_j - \partial_j A_i)}_{F_{ij} = \varepsilon_{ijk}B_k}$$

$$= q\left(E_i + \varepsilon_{ijk}\dot{x}^j B^k\right)$$

$$= q\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right);$$

Lorentz force!