PHYS 5210
Graduate Classical Mechanics
Fall 2023

Lecture 8
Configuration space and Lagrange multipliers

September 15

Configuration space (Q): set of all allowed "positions"

Example 1:

n DOF:

$(x_1, \dots, x_n) \in \mathbb{R}^n$

$$S = \int dt \, \tfrac{1}{2} m \dot{x}_i \dot{x}_i$$

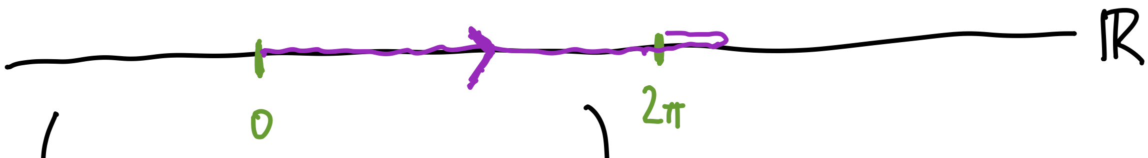
$$Q = \mathbb{R}^n$$

Example 2: pendulum.

$$S = \int dt \, \tfrac{1}{2} I \dot{\theta}^2 + \dots$$

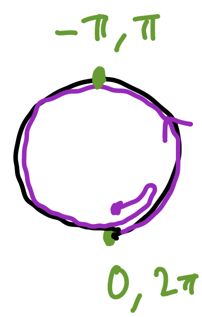
points are identical

$\theta \sim \theta + 2\pi$ $\qquad Q =$ circle (S^1)



$\theta \sim \theta + 2\pi$

$$S^1 = \mathbb{R}/(2\pi\mathbb{Z})$$



Configuration Space = manifold

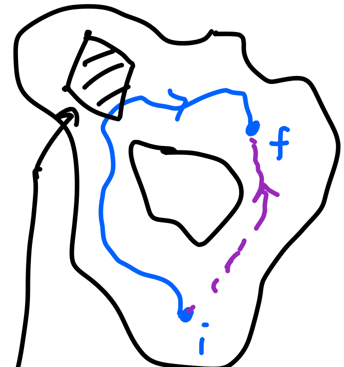
n DOF

$\hookrightarrow$ n-dimensional con. sp.

Principle of least action generalizes:

- $S : \left\{ \begin{array}{l} \text{trajectories on } Q \\ \text{w/ fixed endpoints} \end{array} \right\} \to \mathbb{R}$

- Physical trajectories extremize S

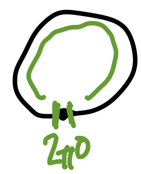


Problem: given proposed $q_i(t)$ $(i = 1, \ldots, n)$

how to evaluate $\dfrac{\delta S}{\delta q_i(t)} = ?$

try to evaluate S in some local coord.

Strategy 1: "guess" coordinates:



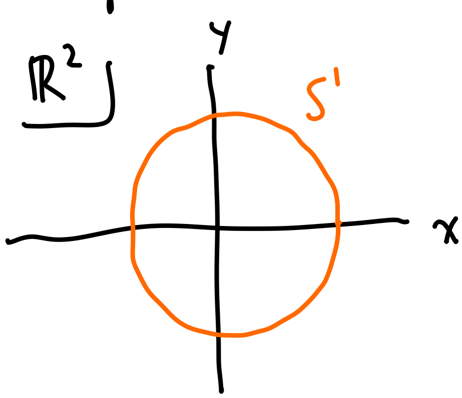
Choose $\theta \in (0, 2\pi)$ $\in \mathbb{R}$ (makes sense)

Demand invariant BBs under $\theta \to \theta + 2\pi$:

$L(\dot\theta, \ldots, \sin\theta, \cos\theta, \ldots)$ $\qquad \dfrac{d\theta}{dt} \to \dfrac{d\theta}{dt} + \cancel{\dfrac{d}{dt} 2\pi}$

Strategy 2: Lagrange multipliers: embed Q into $\mathbb{R}^m$ for $m > n$ (if Q n-dimensional):

Example: $S^1 \subset \mathbb{R}^2$



$\underbrace{x^2 + y^2 - 1}_{f} = 0$

$f = 0$

↳ How to enforce?

Idea: Write $S[\underbrace{x(t), y(t)}_{\mathbb{R}^2}, \lambda(t)]$ ← Lagrange multiplier

$$S = \int dt \left[\frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \cdots + \lambda(x^2 + y^2 - 1) \right]$$

Treat x, y, λ as DOF:

$$\frac{\delta S}{\delta \lambda} = 0 = x^2 + y^2 - 1 \qquad \text{(enforce constraint)}$$

$$\frac{\delta S}{\delta x} = -m\ddot{x} + 2\lambda x = 0 \qquad \frac{\delta S}{\delta y} = -m\ddot{y} + 2\lambda y = 0.$$

Get rid of Lagrange multiplier?

$$x(-m\ddot{x} + 2\lambda x) + y(-m\ddot{y} + 2\lambda y) = 2\lambda \overset{x^2+y^2=1}{\downarrow} 1 - m(x\ddot{x} + y\ddot{y}) = 0.$$

$$\lambda = \frac{m}{2}(x\ddot{x} + y\ddot{y})$$

$$\lambda = -\frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

$$x^2 + y^2 = 1$$

$$\frac{d}{dt} \hookrightarrow x\dot{x} + y\dot{y} = 0$$

$$\frac{d}{dt} \hookrightarrow x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0$$

Hence:

$$\ddot{x} = \frac{2\lambda}{m}x = -(\dot{x}^2 + \dot{y}^2)x$$

$$\ddot{y} = -(\dot{x}^2 + \dot{y}^2)y$$

constraint (normal) forces!

Interpretation: $\lambda \propto$ normal forces.

Strategy 3: (hybrid) Embed $x^2+y^2-1=0$ but solve constraint before POLA:

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + \cdots + \lambda\underbrace{(x^2+y^2-1)} + V(x,y)$$

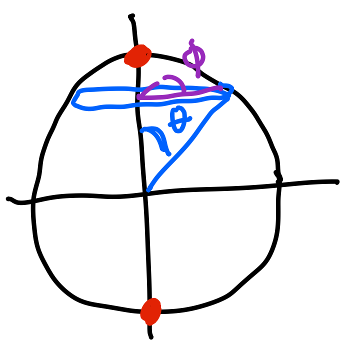
$x^2+y^2=1$ solved by $\boxed{\begin{array}{l} x=\cos\theta \\ y=\sin\theta \end{array}}$

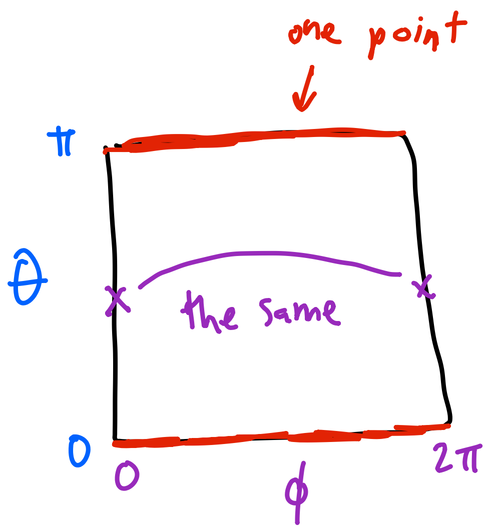
$$L = \frac{m}{2}\left[(-\sin\theta\,\dot{\theta})^2 + (\cos\theta\,\dot{\theta})^2\right] = \frac{m}{2}\dot{\theta}^2 + V(\cos\theta, \sin\theta)$$

Agrees w/ invariant BBs from Strat 1.

Two-dimensional sphere S^2:

Strategy 1:





Strategy 3: $S^2 \subset \mathbb{R}^3$: $\underbrace{x^2+y^2}+z^2=1$

$x^2+y^2 = \sin^2\theta$

$z^2 = \cos^2\theta$

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\downarrow$$

$$= \frac{m}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$