PHYS 5210
Graduate Classical Mechanics
Fall 2023

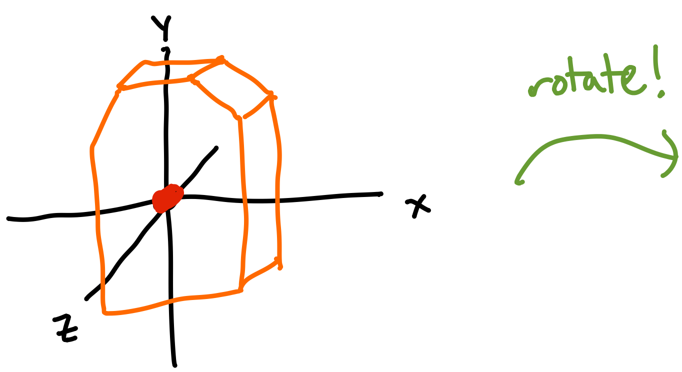
Lecture 9
Configuration space of rigid body rotation

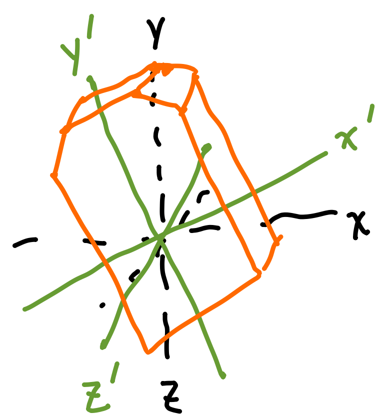
September 18

Configuration space (Q): set of all physically distinct configurations

Today: rigid body rotation with <u>one</u> <u>point</u> fixed

What is Q?





$(x,y,z) = x_i$ (space frame)

$(x',y',z') = x'_I$ (body frame)

$Q = \{\text{rotations of 3-dim space}\}$

$= \{\text{body frame rel. to space frame}\}$

Parameterize body frame? orthonormal basis vector

$$\vec{e}_I = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

x' y' z'

$I=1$ $I=2$ $I=3$

$$\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3) = 1$$

Orthonormal: $\vec{e}_I \cdot \vec{e}_J = \delta_{IJ} = \begin{cases} 1 & I=J \\ 0 & I \neq J \end{cases}$. AND: $\vec{e}_I \cdot (\vec{e}_J \times \vec{e}_K) = \varepsilon_{IJK}$

Also for space frame: $\vec{R}_i = \{\vec{R}_1, \vec{R}_2, \vec{R}_3\}$

$$\vec{R}_i \cdot \vec{R}_j = \delta_{ij} \qquad \vec{R}_i \cdot (\vec{R}_j \times \vec{R}_k) = \varepsilon_{ijk}$$

Note: upper case → body ; lower case → space.

$$Q = \{ \text{all } \vec{R}_1, \vec{R}_2, \vec{R}_3, \ \text{orthonormal} \}$$

how to enforce?

Collect 9 components: $\vec{R}_i \cdot \vec{e}_I = R_{iI}$ ← 3x3 matrix

R_{iI} constrained by orthonormal: $\underbrace{\vec{R}_i \cdot \vec{R}_j}$ $= \delta_{ij}$ $\quad RR^T = 1$ (identity)

$$= \sum_{I=1}^{3} R_{iI} R_{jI} = \overbrace{R_{iI} R_{jI} = \delta_{ij}}^{RR^T = 1}$$

$$Q \to O(3) = \{ R_{iI} : R \text{ orthogonal } (R_{iI} R_{jI} = \delta_{ij}) \}$$

$\det(RR^T) = \det(1)$

$\det(R)^2 = 1$

So

$\det(R) = \pm 1$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad R_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\det(R_1) = 1$ $\qquad$ $\det(R_{-1}) = -1$

If $R(t)$ is continuous, $\det(R)$ can't jump from $1 \to -1$.

$\underline{\det(R) = \text{const.} = +1}$

Classical trajectories for rigid body:

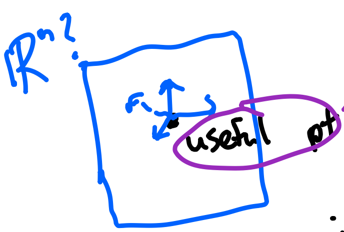
$$Q = SO(3) = \{ R \in O(3) : \det(R) = 1 \}$$
$$= \{ R_{iI} : R_{iI} R_{jI} = \delta_{ij},\ \varepsilon_{ijk} = R_{iI} R_{jJ} R_{kK} \varepsilon_{IJK} \}$$

Configuration space = Lie group. (HW 4 Prob. 4)

How many DOFs? What is dimension of configuration space?

dim(SO(3)) ?

dimension same anywhere



$$R^{(0)}_{iI} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{iI} \in SO(3)$$

What does SO(3) look like near δ_{iI}?

infinitesimal

$$R_{iI} = \delta_{iI} + \varepsilon_{iI}$$

neglect ε^2

$$\hookrightarrow \cancel{\delta_{ij}} = (\delta_{iI} + \varepsilon_{iI})(\delta_{jI} + \varepsilon_{jI})$$
$$= \delta_{iI}\delta_{jI} + \delta_{iI}\varepsilon_{jI} + \varepsilon_{iI}\delta_{jI} + \dots$$
$$= \cancel{\delta_{ij}} + \varepsilon_{ji} + \varepsilon_{ij}$$

$$0 = \varepsilon_{ij} + \varepsilon_{ji}$$ or, ε is antisymmetric

$$\varepsilon_{iI} \rightarrow \begin{pmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}$$ has 3 independent DoF

$\downarrow$

$\dim(SO(3)) = 3.$

Note: $R_{iI} R_{jI} = \delta_{ij}$

Also: $R_{iI} R_{iJ} = \delta_{IJ}$

$RR^T = 1 \rightsquigarrow R$ is invertible

$\cancel{R} R^T R = \cancel{R}$

$R^T R = 1$

$$L = f(\dot{R}_{iI}, R_{iI}, \dots) + \Lambda_{IJ}\underline{(R_{iI}R_{iJ} - \delta_{IJ})}$$

Lagrange multiplier should constrain $R \to$ orthogonal ...

Count: $\dim(SO(3)) = 3.$ $\qquad \dim(\Lambda) = \dim(\mathbb{R}^{3\times 3}) = 9?$

$\dim(R) =$ $\qquad$ 3×3 matrices

How do we only remove 6 DOF?

$R_{iI}R_{iJ} - \delta_{IJ} = R_{iJ}R_{iI} - \delta_{JI}$: symmetric $I \Leftrightarrow J$.

$$\Lambda_{IJ}[\text{symmetric}]_{IJ} = \frac{\Lambda_{IJ}}{2}\underbrace{[\text{sym}]_{IJ} + \frac{\Lambda_{JI}}{2}[\text{sym}]_{JI}}_{=}$$

$$= \underline{\left(\frac{\Lambda_{IJ} + \Lambda_{JI}}{2}\right)}[\text{sym}]_{IJ}$$

only symmetric part of Λ physical

Λ must be symmetric

$$\dim(\text{symmetric } 3\times 3 \text{ matrices}) = \dim(3\times 3) - \dim(\text{antisym } 3\times 3)$$
$$= 9 - 3$$
$$= 6$$