Exam

Due: December 16 at 11:59 PM. Submit on Canvas.

You are allowed to refer to any course materials (including posted solutions), any books, and the Internet (e.g. Wikipedia, papers). **Do not collaborate** with any human or AI; **do not solicit help** via PhysicsForums, Chegg, Quora or any similar website. You may ask the instructor alone for help in the form of clarifying questions. Please cite (in any reasonable way) any references, including online resources, that you have used.

Problem 1 (Particle on a torus): A two-dimensional torus, parameterized by a pair of angle coordinates (θ, ϕ), can be embedded into our three-dimensional world $\mathbb{R}^3$ via the following map:

$$x = (b + a\sin\theta)\cos\phi, \tag{1a}$$

$$y = (b + a\sin\theta)\sin\phi, \tag{1b}$$

$$z = a\cos\theta. \tag{1c}$$

We take the parameters $b > a > 0$.

10 **A:** Following Lecture 8, one way we could deduce the Lagrangian for a non-relativistic particle moving on this torus is to begin with

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \tag{2}$$

and simply plug in the transformation (1). Do this, and find the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$.

10 **B:** Perform a Legendre transform and deduce a Hamiltonian with canonical coordinates $(\theta, \phi, p_\theta, p_\phi)$:

$$H = \frac{p_\theta^2}{2ma^2} + \frac{p_\phi^2}{2m(b + a\sin\theta)^2}. \tag{3}$$

What is the resulting phase space?

10 **C:** Describe qualitatively the possible kinds of trajectories of particles as they move along the torus.

15 **D:** Describe how to solve for the dynamics of the particle using the Hamilton-Jacobi equation. You do not need to evaluate integrals to find the full expression for S.

Problem 2 (Charge density waves): In some solid-state systems, the electron density spontaneously forms a periodic pattern called a **charge density wave**. We can model this in a cartoon effective field theory, where $\phi(x,t)$ denotes e.g. the modulating density variable. Assume that the theory has spacetime translation symmetry, time-reversal symmetry, and lives in one spatial dimension (plus time). Moreover, suppose that for any constant x_0 – and *fixed* constant q – the equations of motion admit 'stable' equilibria

$$\phi_{\mathrm{eq}}(x,t) = \cos(q(x-x_0)). \tag{4}$$

10 **A:** Explain why the simplest effective field theory you can build with all of the necessary symmetries, and which should exhibit stability for configurations (4), has Lagrangian

$$\mathcal{L} = \frac{1}{2}A(\partial_t\phi)^2 - \frac{1}{2}B\left((\partial_x\phi)^2 + q^2\phi^2 - q^2\right)^2. \tag{5}$$

15 **B:** Plug in the ansatz

$$\phi = \cos(qx + \theta(x,t)) \tag{6}$$

into $S = \int \mathrm{d}t\mathrm{d}x\mathcal{L}$. Argue that if $|\partial_x\theta| \ll q$, you find an effective Lagrangian[1]

$$\mathcal{L}_{\mathrm{eff}} = \frac{A}{4}(\partial_t\theta)^2 - \frac{3Bq^2}{4}(\partial_x\theta)^2 + \cdots. \tag{7}$$

Justify all of the (obvious?) continuous symmetries of the new effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$. Deduce the normal modes of the charge density wave effective field theory.

Problem 3 (Modified time-translation symmetry): In this problem, we will build a Lagrangian effective theory for a particle in configuration space $\mathbb{R}^2$, with coordinates (x,y), assuming the modified time-translation symmetry (using the notation of Lectures 2-4):

$$T = 1, \tag{8a}$$
$$X = \alpha y, \tag{8b}$$
$$Y = -\alpha x. \tag{8c}$$

10 **A:** Write down the most general invariant Lagrangian under this continuous symmetry.[2] On very long time scales, what is the effective theory for this system? The answer may depend on unknown functions function of (x,y,t), but you should keep as few time derivatives as possible.

10 **B:** Use Noether's Theorem to find a conservation law associated with this continuous symmetry. Use the leading order effective theory from **A**.

10 **C:** Is it possible to find a Lagrangian L which is not invariant under this continuous symmetry, but for which we would still classify the theory as symmetric? Why or why not?

[1] *Hint:* If θ were constant in x (or very slowly varying...) you can perform the integral over x explicitly to get rid of any trigonometric functions of qx.

[2] *Hint:* Avoid trying to solve the general equation from Lecture 2 for the most general possible L. Instead, first think about physically what this symmetry is, and see if you can leverage some results from class.

Problem 4 (Onset of chaos in the kicked rotor): Consider the kicked rotor (Lecture 36), at perturbatively small ϵ. Work in dimensionless units.

10 **A:** Follow Lecture 32 and aim to develop a perturbation theory for a general discrete map of the form

$$J_{n+1} = J_n + \epsilon K(J_n, \theta_n), \tag{9a}$$
$$\theta_{n+1} = \theta_n + \mathit{\Omega}(J_n) + \epsilon L(J_n, \theta_n) \tag{9b}$$

to first order in ϵ; namely, try to define a

$$\tilde{J}_n = J_n + \epsilon M(J_n, \theta_n) \tag{10}$$

such that

$$\tilde{J}_{n+1} = \tilde{J}_n + \mathrm{O}\left(\epsilon^2\right). \tag{11}$$

Find a formula for such an M by writing both M and K as a Fourier series in the periodic variable $\theta_n \sim \theta_n + 2\pi$. Explain where perturbation theory breaks down at first order.

10 **B:** Now return to the kicked rotor. Where will perturbation theory break down at first order? Analytically predict what the dynamics of the kicked rotor map will look like near the region of phase space where perturbation theory fails. Numerically confirm that your prediction is correct.