Homework 1

Due: September 10 at 11:59 PM. Submit on Canvas.

Problem 1 (Spiral symmetry): Consider a two-dimensional particle moving in a universe with "spiral symmetry", which is invariance under a combined rotation *and* rescaling: in polar coordinates, for fixed constant κ, and rotation angle ϕ, the spiral transformation reads

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{\kappa\phi} r \\ \theta + \phi \end{pmatrix}. \tag{1}$$

You may also assume that the system is time translation invariant.

20 **A:** To deduce the most general effective theory, find the form of 3 invariant building blocks $I(r, \dot{r}, \theta, \dot{\theta})$ under the transformation (1).[1] Show that the following Lagrangian, which has only a single time derivative, is invariant under symmetry (1):

$$L = A(r,\theta)\dot{r}\mathrm{e}^{-\kappa\theta} - U(r,\theta). \tag{2}$$

What are the constraints on the form of A and U?

10 **B:** Apply Noether's Theorem, both for time translation symmetry, as well as for (1). Do you get two independent conserved quantities in general? Why or why not?

10 **C:** Find the most general trajectory that solves the Euler-Lagrange equations. For simplicity, treat $A \neq 0$ as a constant.

10 **D:** In Lecture 1, we described the principle of least action by demanding that S be extremized on physical trajectories, subject to fixing the endpoints of the trajectory. Is it possible to find a trajectory that solves the Euler-Lagrange equations for arbitrary endpoints $r(t_1), r(t_2), \theta(t_1), \theta(t_2)$? If not, what do you think is "problematic" about (2)?

20 **Problem 2 (Translation symmetry, revisited):** In Lecture 3 we remarked that the simplest effective theory with translation invariance has a Lagrangian that does not depend on x. While in physics this is often indeed the case, mathematically it is actually *not* required.

Consider a Lagrangian of the form $L(x, \dot{x})$ with translation (and time-translation) symmetry. Find the most general theory (action) invariant under this transformation, allowing for $\Phi \neq 0$. You should find that it is possible to have $\partial L/\partial x \neq 0$. For such a Lagrangian, explain the physical system it corresponds to and why this system is translation invariant. What does Noether's Theorem for translation symmetry imply for this system?

[1] *Hint:* Try looking for invariant building blocks under only two of the four variables. Which independent pairs should you try?

Problem 3 (Spectral methods): In Lecture 1, we discussed how variational calculus could be "approximated" by considering a discretization of time steps and optimizing over $x(0)$, $x(\Delta t)$, $x(2\Delta t)$, etc. If we wanted to evaluate the principle of least action numerically, this could be one way to proceed. A more clever numerical strategy would be to perform the optimization problem by expressing $x(t)$ in terms of a sum over a finite number of chosen (basis) functions.

We can illustrate the point above using a simple example: consider action

$$S[x(t)] = \int_{-T/2}^{T/2} \mathrm{d}x \left[\frac{m}{2}\dot{x}^2 + \frac{ma^2}{2}x^2\right] \tag{3}$$

where m and a are positive constants. Let us evaluate the principle of least action for trajectories with boundary conditions $x(\pm T/2) = b$, for some constant b.

10 **A:** Solve the Euler-Lagrange equations and impose the desired boundary conditions; conclude that the extremum of the action is

$$x(t) = b\frac{\cosh(at)}{\cosh(aT/2)}. \tag{4}$$

10 **B:** We can exactly reproduce (4) by evaluating the principle of least action over all possible trial functions of the form

$$x(t) = b + \sum_{n=1,3,5,\ldots}^{\infty} c_n \cos\frac{n\pi t}{T}. \tag{5}$$

The variational parameters are the constants c_n; n is restricted to be odd to ensure that the boundary conditions $x(\pm T/2) = b$ are obeyed. Plug in ansatz (5) into (3) and show that[2]

$$S(c_1, c_3, \ldots) = \sum_{n=1,3,5,\ldots}^{\infty} \left[\frac{1}{2}A_n c_n^2 + B_n c_n\right]. \tag{6}$$

Give explicit expressions for A_n and B_n, and use the principle of least action to deduce c_n. An unimportant overall constant offset has been neglected in the formula for S.

10 **C:** Show explicitly that your calculation reproduces (4). To do so, it may be helpful to proceed by writing (4) in the form of (5).[3]

[2] *Hint:* All of the integrals that appear in this calculation should remind you of integrals you would have done when analyzing the infinite square well in quantum mechanics. You can therefore quote the values without justification as appropriate.

[3] *Hint:* Again, this type of calculation should remind you of quantum calculations.

Problem 4 (Symmetry algebras): As we will emphasize more later in the class, it is not possible to impose arbitrary collections of transformations as symmetries on a physical system; symmetries must form a group. In the context of continuous symmetries this has particularly powerful implications in mechanics. Let $x^\mu = (x_i, t)$ denote the spacetime coordinates of a problem, and $X^\mu = (X, T)$ denote the infinitesimal transformations on these coordinates that generate continuous symmetries:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = X^\mu. \tag{7}$$

At finite s, the solution of (7), denoted $x^\mu(s)$, represents a coordinate transformation that must leave the action invariant. An abstract (but useful) notation for this solution is

$$x^\mu(s) = \mathrm{e}^{sX^\mu \partial_\mu} x^\mu \tag{8}$$

where $\partial_\mu = (\partial_i, \partial_t)$ denotes the combined derivatives.

5 **A:** Let $A = A^\mu \partial_\mu$ and $B = B^\mu \partial_\mu$ denote two symmetry transformations. Show that

$$[A, B]^\mu = A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu \tag{9}$$

is also a symmetry generator.[4] This is called the Lie derivative of B along A; such Lie derivatives play an important role in geometric formulations of mechanics.

5 **B:** Now consider Lagrangians of the form $L(x, \dot{x}, t)$ which are invariant under some symmetry algebra with two linearly independent generators A and B. Argue that A and B may always be chosen[5] such that either $[A, B] = 0$, or

$$[A, B] = A. \tag{10}$$

5 **C:** Consider a Lagrangian with symmetry algebra (10) with $A = \partial_t$ corresponding to time translation symmetry. Argue that up to a change of coordinates (which you can assume is not singular, for simplicity), any choice of B which leads to a nontrivial Lagrangian can be written as

$$B = \frac{1}{2} x \partial_x + t \partial_t. \tag{11}$$

Explain physically what this symmetry represents. Find the most general effective theory invariant under the resulting symmetry algebra.

[4] *Hint:* Think about the consistency of the angular momentum algebra in quantum mechanics, and why that underlying structure is required. The notation here is not accidental.

[5] *Hint:* You can add linear combinations of them together and/or multiply by constants.