PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 10

Due: November 12 at 11:59 PM. Submit on Canvas.

Problem 1 (Dying star): A cartoon model for a planet orbiting a dying star is given by the following Hamiltonian:

$$H = \frac{p^2}{2m} - V_0(t) \cdot \begin{cases} 1 - |x|/a & |x| < a \\ 0 & |x| > a \end{cases} . \tag{1}$$

$V_0(t)$ is decreasing as a function of time, corresponding to the gravitational pull of the star decreasing as a function of time. While we could study a more honest model of the dynamics, this toy model will capture the key features of the problem while having simpler algebra.

20 **A:** Assume for the moment that $V_0(t)$ is a constant independent of time. Supposing that $E < 0$, describe the motion of the particle through phase space qualitatively. Construct the action variable J of the appropriate action-angle coordinates, following Lecture 29.

20 **B:** Suppose that at time $t = 0$, the system has energy

$$E(0) = -\frac{3V_0}{8}. \tag{2}$$

For times $0 < t < T$, suppose that

$$V_0(t) = V_0 \left(1 - \frac{t}{2T}\right). \tag{3}$$

Argue that if T is sufficiently large, the particle will necessarily stay bound inside of the potential. Approximately how much energy will it have? How large does T need to be for your conclusions to hold?

10 **C:** Suppose now that $V(t)$ abruptly changes to $V_0/2$ at time $t = 0$. Qualitatively describe the possible dynamics of the system, assuming the same initial energy (2).

Problem 2 (Extra symmetry in an integrable system): Consider a Hamiltonian system on a $2n$-dimensional phase space which is integrable: someone has found action-angle coordinates such that $H = H(J_1, \ldots, J_n)$. Assume that the "angle" coordinates are all periodic and that each frequency $\omega_\alpha = \partial H/\partial J_\alpha \neq 0$.

15 **A:** Suppose that we observe that $\omega_1(\mathbf{J}) = \omega_2(\mathbf{J})$ for all $\mathbf{J}$. Integrability tells us that $\{J_a, H\} = 0$ for $a = 1, \ldots, n$, but argue that you can find one more independent function[1] that also generates a symmetry of H. Conclude that any trajectory through phase space will be dense on at most an $(n-1)$-dimensional subspace.

15 **B:** Suppose that we track a trajectory through phase space, and we find that regardless of which initial conditions we choose, the trajectory seems to fill (in the sense of Lecture 31) an $(n-k)$-dimensional subspace of phase space. Generalize **A** and qualitatively describe the possible additional symmetry generators that the system may have.

[1] *Hint:* Can you write $\omega_1 - \omega_2 = 0$ using Poisson brackets?

Problem 3 (Obstructions to global action-angle coordinates): In many models which we will consider integrable, it is not possible to find a global set of action-angle coordinates. The lack of existence of global action-angle coordinates has interesting implications both from a physics as well as a mathematical perspective.

20 **A:** The simplest example of a system which fails to have global action-angle coordinates is the following Hamiltonian system on phase space $\mathbb{R}^2$, with canonical coordinates:

$$H = p^2 - x^2 + x^4. \tag{4}$$

Sketch the trajectories of this system in phase space and *qualitatively* describe how you would try to find action-angle variables (do not carry out an explicit computation). You should find that there are no global action-angle coordinates. Explain physically why this happens. Give a heuristic argument that the lack of existence of global action-angle coordinates is linked to the existence of a saddle point in the function H at $(x, p) = (0, 0)$.

15 **B:** Take a generic $H(x, p)$ which is not singular (at any point where $\nabla H = 0$, $\partial_i \partial_j H$ is a non-singular matrix with two non-zero eigenvalues), and for which

$$\lim_{\sqrt{x^2+p^2}\to\infty} H = \infty. \tag{5}$$

Suppose that H has s saddle points. Find the maximum and minimum number of action-angle coordinate patches necessary to cover all of phase space $\mathbb{R}^2$, along with examples of Hamiltonian dynamical systems that saturate your upper and lower bound.