PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 11

Due: November 19 at 11:59 PM. Submit on Canvas.

Problem 1 (Disordered box): In this problem, use the perturbation theory for a single action-angle pair developed in Lecture 32.

15 **A:** Consider a problem in which $H = H_0(J_0) + \epsilon H_1(\phi_0, J_0)$ following Lecture 32, and further suppose that the Fourier coefficient $h_0 = 0$. Carry out perturbation theory to second order, starting with writing generator $S = \phi_0 J + \epsilon S_1 + \epsilon^2 S^2 + \cdots$. Show that in this case, the Hamiltonian is given, up to second order, by

$$H(J) = H_0(J) - \frac{\epsilon^2}{2}\frac{\partial}{\partial J}\int_0^{2\pi}\frac{\mathrm{d}\phi_0}{2\pi}\frac{H_1(\phi_0, J)^2}{\omega_0(J)} + \cdots . \tag{1}$$

10 **B:** Suppose H_0 describes a free particle of mass m in a box with hard walls of size L. Argue that in this case, the momentum p becomes proportional to the action variable J, and that

$$H_0(J) = \frac{\pi^2 J^2}{2mL^2}. \tag{2}$$

15 **C:** Now, consider $H = H_0 + \epsilon V(x)$, where H_0 is (2) and $V(x)$ is a random function with zero mean:

$$\int_0^L \mathrm{d}x V(x) = 0. \tag{3}$$

Show that this perturbation obeys the criteria of **A**. Compute the leading-order correction to the frequency of oscillations of a particle in the disordered box. Does the sign of your answer (does the oscillation period increase or decrease?) make physical sense?

25 **Problem 2 (Perturbation theory to all orders):** In this problem we will look at a very simple example where the high order perturbation theory of Lecture 34 can be carried out explicitly. Consider the simple harmonic oscillator, which we solved using action-angle variables in Lecture 29:

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{4}$$

Now consider modifying the Hamiltonian to $H = H_0 + \epsilon H_1$ where

$$H_1 = \epsilon \cdot \frac{1}{2}m\omega^2 x^2. \tag{5}$$

First, explain with as little computation as possible what $H(J;\epsilon)$ is. Your expression should be exact to all orders in ϵ. Then, show how to exactly reproduce this result using the high-order perturbation theory, resummed to all orders.[1]

[1] *Hint:* Start at first order in perturbation theory, and find the form of A_1. Then, argue that $\{A_1, H_0\}$ and $\{A_1, H_1\}$ always reduce to linear combinations of H_0 and H_1. Conclude that *all* A_n can be chosen to be proportional to A_1, and therefore that all you need to do is solve $\exp[s \cdot \mathrm{ad}_{A_1}](H_0 + \epsilon H_1) = H_0$ for the parameter $s(\epsilon)$.

Problem 3 (Billiard ball): Consider a free particle in an approximately circular box of radius a. The Hamiltonian of the system in polar coordinates is

$$H = \frac{p_r^2}{2M} + \frac{p_\theta^2}{2Mr^2} + V_{\text{wall}}(r, \theta) \tag{6}$$

where we will approximate that for some integer $m > 1$,

$$V_{\text{wall}}(r, \theta) = A \cdot \Theta(r - a + \epsilon a \cos(m\theta)) \approx A \cdot \Theta(r - a) + \epsilon a A \cdot \delta(r - a) \cos(m\theta). \tag{7}$$

Here Θ is the step function

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}. \tag{8}$$

while δ is the Dirac δ function. As depicted in Figure 1, this potential is a decent proxy for the shape of the box's walls, with ϵ proportional to the deviation away from circular walls. In the entire problem that follows, you should always work with V_{wall} Taylor expanded to first order in ϵ, for convenience.

15 **A:** First, analyze the problem when $\epsilon = 0$, so the box is perfectly circular. Describe how action-angle variables can be found: one associated with θ and one associated with r, assuming that the energy of the particle is smaller than A. While $H(J_r, J_\theta)$ does not have a nice expression, write down a formal integral expression for J_r as a function of E and J_θ. Differentiate this expression with respect to J_r and J_θ and show that

$$\frac{\omega_\theta}{\omega_r} = \frac{1}{\pi} \arccos \frac{J_\theta}{a\sqrt{2ME}}. \tag{9}$$

10 **B:** Sketch the trajectories of the billiard, when ω_r and ω_θ are commensurate, and incommensurate. In each case, describe what dimensional subspace of the disk, in (r, θ) coordinates, will be explored densely over time. Argue that if $J_\theta = 0$ or $J_\theta = \pm a\sqrt{2mE}$, the action-angle coordinates above are sick, and we should neglect this part of phase space.

10 **C:** Deduce that $m = 0, 1, 2$ perturbations might be integrable everywhere, as there is no obvious breakdown of perturbation theory. What shapes are the walls in these cases?

15 **D:** Describe what happens at first order in perturbation theory, assuming that $m > 2$. In which parts of phase space does perturbation theory break down? Sketch the billiard's trajectories in the deformed circular box as accurately as you can in these regions.

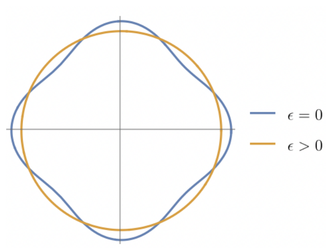

Figure 1: A sketch of the billiard's domain with and without the perturbation ϵ, for the specific case $m = 4$.