Homework 12

Due: December 10 at 11:59 PM. Submit on Canvas.

Problem 1 (Bifurcations): Consider a "generic" one-dimensional map

$$x_{n+1} = f(x_n). \tag{1}$$

When the stability and/or number of fixed points of one of these maps changes abruptly as the function f is perturbed, we say that there is a **bifurcation**. In this problem, we will explore the kinds of generic bifurcations that can arise in one-dimensional maps.

15 **A:** Suppose that $f(x)$ is an increasing function (i.e. $f'(x) > 0$) in the interval $x_1 < x < x_2$, and assume that $f(x_1) \neq x_1$ and $f(x_2) \neq x_2$. Show that[1] if there are k unstable fixed points, there are either $k-1$, k or $k+1$ stable fixed points. Describe how, if you perturbed the function $f(x)$ sufficiently weakly,[2] you could only ever create (or remove) a *pair* of two fixed points – one stable, and one unstable.

15 **B:** Show that if $f(x)$ is allowed to be decreasing as well, then it is possible for a stable fixed point to be perturbed into an unstable fixed point, without the creation of any new fixed point pair. By considering what would happen to the map $f^{[2]}(x)$ from Lecture 38, argue that when a stable fixed point turns into an unstable fixed point, we will generically have a **period doubling bifurcation**, as in Lectures 38 and 39. Your result implies that the period doubling transition to chaos should be generic for one-dimensional maps.

15 **C:** As an explicit example of a perturbed map, consider the perturbed logistic map

$$x_{n+1} = rx_n(1-x_n) + \epsilon \tag{2}$$

for $0 < r < 3.1$ and $\epsilon \geq 0$ sufficiently small. Calculate the location and stability of fixed points as a function of ϵ, within this range of r. Show that when $\epsilon = 0$, the motion of fixed points under the perturbation of increasing r is unusual, and explain why. In contrast, show that for $\epsilon > 0$, the behavior of the map is "typical" and fits into the phenomenology you found above for "generic" maps.[3]

Problem 2 (Quartic Hamiltonian): Consider canonical coordinates on phase space $\mathbb{R}^4$, with Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^4 + y^4 + 2\alpha x^2 y^2}{4}. \tag{3}$$

10 **A:** Take $\alpha \geq 0$. Where can you prove (easily!) the system is integrable?

15 **B:** Numerically look for integrability vs. chaos. Can you find compelling evidence for chaos at certain values of α? (You don't need to exhaustively characterize where there is chaos.)

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[1] *Hint:* Draw a graph of x and (any appropriate function) $f(x)$ in the interval $[x_1, x_2]$. What are the possible choices of $f(x)$ you need to consider?

[2] *Hint:* The perturbed $f(x)$ is still increasing.

[3] Feel free to use numerical methods if necessary to discuss what happens to the period-doubling bifurcation near $r = 3$.

Problem 3 (Solvable limit of chaos in the logistic map): Consider the logistic map of Lecture 37 at the maximal value $r = 4$:

$$x_{n+1} = 4x_n(1 - x_n). \tag{4}$$

10 **A:** Suppose that we write

$$x_n = \frac{1 - \cos(2\pi y_n)}{2} \tag{5}$$

where we identify $y_n \sim y_n + 1$ as a periodically identified variable. Plug (5) into (4) and use trigonometric identities to show that[4]

$$y_{n+1} = 2y_n \ (\mathrm{mod}\ 1) = 2y_n - \lfloor 2y_n \rfloor. \tag{6}$$

10 **B:** Let us now show that (6) is an exactly solvable model of chaos. Without loss of generality, we may write

$$y_0 = \sum_{k=1}^{\infty} 2^{-k} b_k \tag{7}$$

where each $b_k = 0$ or 1. What is y_n? Suppose that we start with two initial conditions y_0 and y_0' obeying $|y_0 - y_0'| = \epsilon \ll 1$. At what step n can $|y_n - y_n'| \sim 1$? Compare the divergence of trajectories to the butterfly effect discussed in Lecture 40, and estimate a "Lyapunov exponent" of the logistic map at $r = 4$ analytically.

10 **C:** Calculate the box dimension (Lecture 41) of the set S, defined by

$$S = \{y_0 \text{ such that for all sufficiently large } n,\ y_n = 0\}. \tag{8}$$

5 **D:** Now that you have seen the solution to the $r = 4$ logistic map, argue that for a typical trajectory not in S, the probability density for x_n (at large n, in the original x coordinates) is

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{9}$$

10 **E:** One way to define a "multifractal dimension" d_q is as follows. Consider ϵ^{-1} boxes of size ϵ, and define

$$\mu_j = \mathbb{P}[(j-1)\epsilon < x_n < j\epsilon] \tag{10}$$

to be the probability that x is found in box j. Define

$$d_q = \lim_{\epsilon \to 0} \frac{1}{1-q} \frac{1}{\log \epsilon^{-1}} \sum_{i=1}^{\epsilon^{-1}} \mu_i^q. \tag{11}$$

Calculate d_q for the $r = 4$ logistic map, and compare your answer to **C**. If you find that this definition of multifractal dimension d_q does detect the fractal structure of the logistic map, explain intuitively what is responsible for the effect.

[4] Here a (mod 1) denotes only the "decimal" part of a, e.g. $4.32(\mathrm{mod}\ 1) = 0.32$.