PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 2

Due: September 17 at 11:59 PM. Submit on Canvas.

Problem 1 (Twisting magnetic field): Consider a nonrelativistic particle of mass m and charge q moving in the presence of the following electromagnetic potential:

$$
\begin{aligned}
A_t &= 0, && (1a)\\
A_x &= -\frac{b}{k}\sin(kz), && (1b)\\
A_y &= -\frac{b}{k}\cos(kz), && (1c)\\
A_z &= 0. && (1d)
\end{aligned}
$$

10 **A:** What are the electromagnetic fields? Sketch the shape of the field lines.

20 **B:** Write down the Lagrangian for the particle. Show that by using continuous symmetries of this Lagrangian, and following the reduction that we did for the central force problem, that you can reduce the dynamics to that of an effective particle moving only in the z direction.

10 **C:** Explain the trajectories of particles for general initial conditions; an analytic solution is not necessary, so long as you can give a clear qualitative description of the possibilities.

20 **Problem 2 (Relativistic particles and tensor fields):** In Lecture 7, we discussed how the relativistic particle must couple, essentially uniquely, to the electromagnetic field. Suppose, however, that the universe was filled instead with some other "tensor field" $K_{\mu\nu}(x) = K_{\nu\mu}(x)$. We demand that the coupling of our particle's worldline $x^\mu(\lambda)$ to $K_{\mu\nu}$ should be covariant (spacetime indices contracted properly), and that, if the field $K_{\mu\nu}$ is constant (independent of x), the action has translation symmetry.

As in Lecture 7, postulate an action

$$S[x] = \int \mathrm{d}\lambda \left[-m\sqrt{-\frac{\mathrm{d}x_\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}} + K_{\mu\nu}(x)\cdot[\cdots]^{\mu\nu} + \cdots \right]. \tag{2}$$

Argue that up to overall scaling constants (like m), the form of $[\cdots]^{\mu\nu}$ above is uniquely fixed by reparameterization symmetry in λ, and Lorentz index contraction. What do the higher order terms in K look like?

Problem 3 (Time-reversal breaking in the 2d harmonic oscillator): Consider a theory of two interacting particles with coordinates x and y. Suppose that the theory is rotation invariant, and that the theory has a stable equilibrium configuration at $x = y = 0$.

10 **A:** Following Lecture 5, and using results about symmetries and invariant building blocks from Lectures 2-4, explain why the most general Lagrangian, up to quadratic order in x and y, is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{\gamma}{2}\left(x\dot{y} - y\dot{x}\right) - \frac{k}{2}\left(x^2 + y^2\right). \tag{3}$$

In the absence of time-reversal symmetry, $\gamma \neq 0$ is allowed. Stability of the theory requires $m, k > 0$.

15 **B:** Solve the Euler-Lagrange equations and find the normal mode frequencies of this oscillator.[1]

15 **C:** If we had wanted to build an effective theory whose regime of validity described dynamics on long time scales, we might have first set $m = 0$. What would the normal modes have been in that case? Compare to the general solution in **B**, and deduce whether there are any limits in which your answer approximately agrees. If such a limit agrees, sketch what happens in the full theory of **B**, together with the approximation made by $m \approx 0$. Does your answer make physical sense?

One setting where this model is relevant is to the dynamics of charged particles in a magnetic field perpendicular to the plane, in which case $\gamma = qB$ where B is the strength of the magnetic field, The type of dynamics you should have found in **C** is relevant to the motion of light particles (e.g. electrons) in plasmas in strong magnetic fields.

15 **Problem 4 (Expanding universe):** Consider a particle moving in one dimension, in a universe with space translation symmetry but not time translation symmetry, which we assume has been broken by (e.g.) the Big Bang. In this universe, it could be reasonable to replace the Lorentz boost symmetry with one generated by[2]

$$X = \frac{x^2}{2} + \frac{c^2}{2H^2}\mathrm{e}^{-2Ht}, \tag{4a}$$
$$T = -\frac{x}{H}. \tag{4b}$$

Here H is a constant with units of inverse time, called the **Hubble constant**, and c is the speed of light. This turns out to be a symmetry in an expanding universe, such as the one we think we live in.

Find the most general effective theory compatible with the continuous symmetry (4), together with spatial translation $(X, T) = (1, 0)$, which also has the property that as $Ht \to 0$, it can reproduce the nonrelativistic mechanics of our everyday universe. Describe the trajectory of a particle in this universe for generic initial conditions.

[1] *Hint:* You may find it useful to set $z = x + \mathrm{i}y$, and look for solutions where $z(t) = z_0\mathrm{e}^{-\mathrm{i}\omega t}$ where z_0 is some complex coefficient.

[2] I don't think it's obvious why this works, but just follow along until the end and the final answer will indeed reproduce the equations of motion of a particle in an expanding universe!