Homework 3

Due: September 24 at 11:59 PM. Submit on Canvas.

Problem 1 (Two rods): Consider the cartoon system sketched in Figure 1.

10 **A:** What is the configuration space? In your answer, you can assume that the rods are perfectly rigid and neglect their internal dynamics.

15 **B:** Suppose that the theory has time-translation symmetry, time-reversal symmetry, and a global rotation symmetry in the plane. Write down the most general Lagrangian (up to total derivative terms), with up to two time derivatives which is invariant under all of the symmetries. Be sure to explain why your answer is reasonable, given the configuration space. Feel free to use the coordiantes in Figure 1.

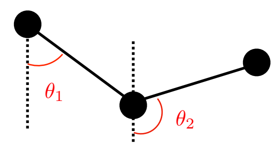

Figure 1: Two rigid rods are attached to each other, such that the only allowed motion is the relative rotation of the rods around the left (fixed) point.

15 **C:** Now, propose an effective theory for the dynamics of this composite object, whose regime of validity is long time scales compared to *all* "internal" dynamics of the object. What is the configuration space of this effective theory, and what symmetries does the new effective theory have?[1]

Problem 2 (Translational dynamics of the rigid body): In Lectures 9 and 10, we described the motion of a rigid body assuming that the only dynamics corresponded to the overall rotation of the object. In this problem, we consider a rotating object whose center of mass can also freely move. The configuration space will then be $\mathbb{R}^3 \times \mathrm{SO}(3)$, with coordinate z_i corresponding to the center of mass coordinate, and orthogonal matrix R_{iJ} corresponding to the orientation of the object.

Modeling the motion of objects in our actual universe, we wish to impose a number of symmetries on the Lagrangian. (*1*) Assume that we have left-SO(3) invariance, which translates $R_{iJ} \to Q_{ij}R_{jJ}$ and $z_i \to Q_{ij}z_j$ for an arbitrary orthogonal matrix Q_{ij}. (*2*) Assume we have time-translation symmetry, along with spatial translation symmetry $z_i \to z_i + a_i$ for any constant vector a_i.

20 **A:** Argue that the most general possible symmetric Lagrangian that depends on ≤ 2 time derivatives, is

$$L = \frac{1}{2}K_{IJ}\dot{R}_{iI}\dot{R}_{iJ} + \frac{m}{2}\dot{z}_i\dot{z}_i + b_I\dot{z}_i\dot{R}_{iI} + c_I\dot{z}_iR_{iI} + A_{IJ}R_{iI}\dot{R}_{iJ} + \frac{1}{2}\Lambda_{IJ}\left(R_{iI}R_{iJ} - \delta_{IJ}\right). \tag{1}$$

Explain how a "clever" adjustment to coordinates/other parameters can be used to assume $b_I = 0$.

25 **B:** When time-reversal symmetry is broken, $A_{IJ} = -A_{JI} \neq 0$ and $c_I \neq 0$ are possible. Follow Lecture 10, and derive a generalization of the Euler equations. Show that if you introduce a Noether charge p_i for translation, the modified Euler equations only depend on R_{iI}. Discuss the simplifications that occur if $c_I = 0$, and compare to the standard Euler equations.

[1] *Hint:* Think about the example of the diatomic molecule in Lecture 5.

Now consider a microscopic model of the rigid body, which has configuration space $\mathbb{R}^{3n}$, where n is the number of particles in the system. Let $x_{i\alpha}$ denote the coordinates where $i = 1, 2, 3$ and $\alpha = 1, \dots, n$ denote the space frame index and particle number, respectively.

15 **C:** Argue that the most general Lagrangian with the necessary symmetries is

$$L = \frac{1}{2} M_{\alpha\beta} \dot{x}_{i\alpha} \dot{x}_{i\beta} + N_{\alpha\beta} x_{i\alpha} \dot{x}_{i\beta} - V\left((x_{i\alpha} - x_{i\beta})(x_{i\gamma} - x_{i\delta})\right) \tag{2}$$

where $M_{\alpha\beta} = M_{\beta\alpha}$ and $N_{\alpha\beta} = -N_{\beta\alpha}$ could in principle depend on the same invariant building blocks as V. For simplicity, you may assume $M_{\alpha\beta}$ and $N_{\alpha\beta}$ do not depend on the coordinates $x_{i\alpha}$. Argue in one or two sentences that in an effective theory for the slow dynamics, we can approximate that V takes its minimum value, which implies that

$$x_{i\alpha}(t) = z_i(t) + R_{iJ}(t) \bar{x}_{J\alpha} \tag{3}$$

where $\bar{x}_{J\alpha}$ are fixed constants that correspond to an equilibrium configuration of V. This change reduces the configuration space to $\mathbb{R}^3 \times \mathrm{SO}(3)$, as before. Argue that without loss of generality, you may choose

$$\sum_{\alpha,\beta} \bar{x}_{J\alpha} M_{\alpha\beta} = 0. \tag{4}$$

Plug in this ansatz into (2), and thus explain the choice $b_I = 0$; otherwise, show that you reproduce (1), providing expressions for the phenomenological parameters in terms of M and N.

20 **Problem 3 (Liquid droplet):** In this problem, we will consider a cartoon model for a fluid droplet in two spatial dimensions as a problem with an interesting configuration space. For simplicity, we will restrict to deformations of the droplet of the form

$$x^i(t) = M^i{}_I(t) x^I, \tag{5}$$

where $M^i{}_I(t)$ is an invertible 2×2 matrix. Our key postulates will be that; (*1*) the fluid can freely rotate or shear in static equilibrium, but that it cannot change its volume while staying in equilibrium, and (*2*) the theory is invariant under arbitrary global invertible transformations in the space frame.

Assume time-reversal symmetry. Argue that the most general Lagrangian describing the droplet, up to lowest nontrivial order in derivatives, is[2]

$$L = \frac{1}{2} A^{IJ}_{KL} \dot{M}^i_I M^K_i \dot{M}^j_J M^L_j - V\left(\epsilon^{ij}\epsilon_{IJ} M^i_I M^j_J\right) \tag{6}$$

where M^I_i is the inverse of M^i_I: $M^i_I M^I_j = \delta^i_j$ and $M^I_i M^i_J = \delta^I_J$, and

$$\epsilon^{ij}\epsilon_{IJ} M^i_I M^j_J = \det M; \tag{7}$$

you don't need to show (7). Argue in one or two sentences that if we build an effective theory of the slowest degrees of freedom, it has configuration space X, given by the space of all invertible 2×2 matrices with determinant 1:

$$X = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } \det(M) = ad - bc = 1 \right\} = \mathrm{SL}(2, \mathbb{R}). \tag{8}$$

$\mathrm{SL}(2, \mathbb{R})$ forms a group of matrices; you don't have to show this.[3] We can use Lagrange multiplier λ to build an effective theory on configuration space X:

$$L = \frac{1}{2} A^{IJ}_{KL} \dot{M}^i_I M^K_i \dot{M}^j_J M^L_j + \lambda\left(\epsilon^{ij}\epsilon_{IJ} M^i_I M^j_J - 1\right) \tag{9}$$

For simplicity we've just set $A = 1$. Here $\epsilon_{xx} = \epsilon_{yy} = 0$ and $\epsilon_{xy} = -\epsilon_{yx} = 1$. You can take as given (or check if you like) that this Lagrange multiplier enforces $\det M = 1$. Find the equation of motion for M^i_I, and comment on the result.

---

[2] *Hint:* The only dependence on M, not its derivatives, should come from transformations which change the volume of the droplet. How does the volume of an infinitesimal bit of the droplet change under a linear transformation (5)?

[3] Strictly speaking, the configuration space is the part of $\mathrm{SL}(2, \mathbb{R})$ that is continuously connected to the identity matrix, but you don't need to worry about that.