PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 4

Due: October 1 at 11:59 PM. Submit on Canvas.

Problem 1 (Vibrations of plates): In this problem, we consider the transverse motion of a thin plate or membrane of width w. Although we will consider the theory of solids more carefully later in the class, we can already use our effective field theory techniques to make testable physical predictions for the motion of the thin plate. The most apparent (in experiment!) motion of the thin plate would be "flexural" motion of the plate transverse to the plane. Let $Z(x, y)$ denote such a vertical displacement of the plate away from coordinate (x, y) in the plane (assuming $Z = 0$ corresponds to the plate at rest). When building the effective field theory Lagrangian, assume spacetime translation invariance, and inversion symmetries under flipping the sign of any coordinate (e.g. $t \to -t$).

15 **A:** Begin by considering a rigid solid plate. Explain what (reasonable) physical symmetries require that the following configurations represent static equilibria of the plate, for any constants $a_{1,2,3}$:

$$Z_{\mathrm{eq}}(x, y) = a_1 + a_2 x + a_3 y. \tag{1}$$

Conclude that the most general $\mathcal{L}$ for which $\mathcal{L}(Z) = \mathcal{L}(Z + Z_{\mathrm{eq}})$ is

$$\mathcal{L} = A(\partial_t Z)^2 - B\partial_i\partial_j Z \partial_i\partial_j Z + \cdots, \tag{2}$$

where we have kept the lowest nontrivial terms in both space and time derivatives, and not written down terms that are (related to existing terms by) a total derivative. On what length scales do you expect this effective field theory to make sense?

15 **B:** Follow the derivation of the Euler-Lagrange equations for field theory in Lecture 12, and show that if $\mu = t, x, y$ represent the spacetime coordinates,[1]

$$\frac{\delta S}{\delta Z} = \frac{\partial \mathcal{L}}{\partial Z} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu Z)} + \partial_\nu\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\nu\partial_\mu Z)} \tag{3}$$

10 **C:** Given the theory in (2), evaluate the Euler-Lagrange equations. Find the dispersion relation of the normal modes $Z \sim \mathrm{e}^{\mathrm{i}k_j x_j - \mathrm{i}\omega t}$.

10 **D:** Later in the class, we will derive that (multiple types of) sound waves propagate in bulk solids with a characteristic speed of sound $c \sim 3000$ m/s. Use dimensional analysis, as well as the effective theory above, to estimate the typical frequency of vibrations of a solid plate of diameter $r = 10$ cm, with thickness $w = 1$ mm. Would such sound waves be audible to the human ear?

15 **E:** Argue that if the solid plate is held under tension at its boundary, then in (1) we should force $a_2 = a_3 = 0$. What is the resulting change to the effective field theory $\mathcal{L}$, and what is the corresponding change to the normal mode dispersion? On what length scales is the resulting change the most drastic?

[1] *Hint:* You'll need to modify the boundary conditions in the principle of least action, fixing both Z and $\partial_\mu Z$.

Problem 2 (Broken left-SO(3) invariance): In this problem we will study the dynamics of a rigid body with broken left-SO(3) invariance. One example is a system described by the Lagrangian

$$L = \frac{1}{2}K_{IJ}\dot{R}_{iI}\dot{R}_{iJ} - \frac{1}{2}Q_{IJ}S_{ij}R_{iI}R_{jJ} + \mathit{\Lambda}_{IJ}\left(R_{iI}R_{iJ} - \delta_{IJ}\right), \tag{4}$$

with $K_{IJ} = K_{JI}$, $Q_{IJ} = Q_{JI}$ and $S_{ij} = S_{ji}$.

15 **A:** Show that if you define $T_{IJ} = R_{iI}R_{jJ}S_{ij}$, the Euler-Lagrange equations for R_{iI} can be converted into the generalized Euler equations:[2]

$$\dot{\mathsf{L}} = [\mathsf{L}, \mathit{\Omega}] + [T, Q], \tag{5a}$$

$$\dot{T} = [T, \mathit{\Omega}]. \tag{5b}$$

Here $\mathsf{L} = K\mathit{\Omega} + \mathit{\Omega}K$ is the angular momentum matrix from Lecture 10.

20 **B:** These equations do not admit a general solution, so we make a number of simplifications to proceed. First, assume that $K_{IJ} = K_0\delta_{IJ}$. Second, assume that you have a right-SO(2) symmetry corresponding to rotation around a particular axis oriented in the direction of unit vector n_I in the body frame. Argue that this then implies that, without loss of generality, you may take

$$Q_{IJ} = n_I n_J. \tag{6}$$

Argue that you may also assume, without further loss of generality, that S is diagonal. By writing down (4) in the Euler angle coordinates, explain how, if the particle is not rotating around the n_J axis at time $t = 0$, then its dynamics can be reduced to a particle moving on S^2 with a specific Lagrangian (which you should derive).[3]

Problem 3 (Deformable blob): Consider an initially spherical blob made out of a constant density material. If the blob stretches (or compresses) around the 3-axis in the body frame by a factor of λ, then to compensate, there is a compression (or stretching) of the body in the 1/2-directions in the body frame, such that the net volume of the blob stays constant at any time.

10 **A:** If the spherical blob has moments of inertia $I_{1,2,3} = I_0$, deduce $I_1(\lambda) = I_2(\lambda)$ and $I_3(\lambda)$.[4]

10 **B:** Now, the blob undergoes free rigid body rotation. Describe how the blob should distort itself – namely, describe what $\lambda(t)$ ought to do – if it wants to minimize its kinetic energy at late times. You should assume that the blob starts with a generic initial condition, including a generic value of $\lambda(t = 0)$; your answer might depend on these initial conditions.

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[2] *Hint:* Don't repeat all the steps from Lecture 10, just describe how the Q-dependent term would "follow through" the rest of the manipulations.

[3] You do not need to calculate the Euler-Lagrange equations for the system on S^2; stop once you arrive at the Lagrangian.

[4] *Hint:* There is a very elegant argument which does not require evaluating any integrals.