Homework 5

Due: October 8 at 11:59 PM. Submit on Canvas.

Problem 1 (Three phases of matter): In this problem, we will explore a number of different effective field theories for phases of matter with a U(1) symmetry. We begin with a "microscopic model": a relativistic field theory of a single complex-valued scalar field ϕ (Lecture 16) with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi + \frac{r}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4. \tag{1}$$

In this problem, we suppose that $r, \lambda > 0$ (opposite sign convention on r, vs. Lecture 16).

10 **A:** When $r > 0$, explain why $\phi = 0$ corresponds to an unstable equilibrium. To do so, you should describe the normal modes when linearizing the equations of motion around equilibrium.

20 **B:** The better equilibrium solution is given by

$$\phi = \phi_0 > 0, \tag{2}$$

for an appropriate constant ϕ_0, which you should explain. Show that the solution (2) is not invariant under the U(1) symmetry. We then say that the state **spontaneously breaks** U(1). This phase of matter corresponds to a **superfluid**. We can explicitly deduce the effective field theory for superfluidity by writing

$$\phi = a\mathrm{e}^{\mathrm{i}\theta} \tag{3}$$

in terms of real fields a and θ. Plug (3) into (1), and obtain the quadratic Lagrangian in fluctuations around the point $a = \phi_0 + \delta a$ and $\theta = 0 + \delta\theta$. Describe the normal modes and confirm the stability of this equilibrium. Argue that on long time scales, δa would be a fast and ignorable degree of freedom; the effective field theory of the superfluid becomes

$$\mathcal{L} = -\frac{1}{2}\phi_0^2 \partial_\mu \theta \partial^\mu \theta. \tag{4}$$

15 **C:** How does the U(1) symmetry act on the field(s)? Show that $\mathcal{L}$ is still U(1)-invariant. Find the Noether current for the U(1) symmetry, using the superfluid effective theory (4). If you want to describe a state at a spatially uniform charge density ρ of this conserved U(1), what field configuration state θ should you choose?

15 **D:** If the U(1) symmetry is unbroken, and the theory remains at finite density, it turns out that you should impose the stronger symmetry

$$\theta \to \theta + f(x_i) \tag{5}$$

on the effective field theory for the resulting **normal fluid** phase. Here x_i denotes the spatial coordinates only. While in general this makes the effective field theory rather boring, if we consider a theory in $1+1$ spacetime dimensions, explain why the following Lagrangian is symmetric:

$$\mathcal{L} = -P(\partial_t\theta) + c\partial_t\theta\partial_x\theta. \tag{6}$$

for any function P and constant c. c is called the coefficient of the **chiral anomaly**. Expanding P to quadratic order, find the normal modes and comment on the result.

The anomalous phase cannot be derived from the Lagrangian (1). In condensed matter physics, one often thinks of this anomaly as signaling that the theory is the "edge theory" of a higher-dimensional theory – for example, the chiral edge theory of a quantum Hall state (two-dimensional electron gas in a strong magnetic field).

20 **E:** As in Lecture 16, let us now consider the gauged version of the U(1)-symmetric theory. Adding the gauge-invariant $F^2/4$ term to $\mathcal{L}$, this will describe a **superconducting** phase of matter. Argue that the effective field theory for this superconductor is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\phi_0^2\left(A_\mu - \partial_\mu\theta\right)\left(A^\mu - \partial^\mu\theta\right) \tag{7}$$

Show that using the change of variable

$$A_\mu \to A_\mu + \partial_\mu\theta, \tag{8}$$

the Lagrangian becomes independent of θ, but picks up a "mass term" for A_μ which appears to break gauge invariance. This is called the **Higgs mechanism**. Find the normal modes and compare to those of ordinary electromagnetism.

Problem 2 (Subsystem symmetry): This problem explores a toy classical model for a kind of "fracton matter", which has been a subject of much recent interest in the literature. These systems have unusual symmetries which give rise to exotic effective field theories, one of which you can construct in this problem.

20 **A:** Consider a two-dimensional square lattice with variables θ_{xy} defined at lattice site (x,y), which we assume take on integer values. Consider the microscopic Lagrangian

$$L = \sum_{x,y}\left[\frac{A}{2}\dot{\theta}_{xy}^2 + B\cos\left(\theta_{xy} - \theta_{(x+1)y} - \theta_{x(y+1)} + \theta_{(x+1)(y+1)}\right)\right] \tag{9}$$

Follow Lecture 12 and construct an effective field theory for a field, $\theta(x,y)$, to lowest non-vanishing order in derivatives. Find the normal mode dispersion relation.[1]

[1] *Hint:* Recall the Euler-Lagrange equations for a higher derivative Lagrangian from Homework 4.

15 **B:** Your effective field theory should be invariant under

$$\theta \to \theta + f(x) + g(y) \tag{10}$$

for arbitrary functions f and g – this is called a U(1) **subsystem symmetry**. Show that the equation of motion for θ in *any* effective field theory, in two spatial dimensions, whose Lagrangian $\mathcal{L}(\theta, \ldots)$ is invariant under (10) must take the form:

$$\partial_t \rho + \partial_x \partial_y J_{xy} = 0, \tag{11}$$

where ρ and J_{xy} depend on θ (and any other fields that might be present in the EFT). Explain how this equation implies that not only is

$$Q = \int \mathrm{d}x\mathrm{d}y\ \rho \tag{12}$$

a constant of motion, but there exist infinitely many other constants of motion as well. Give expressions for the other conservation laws as well as a simple physical explanation for what they correspond to.