Homework 6

Due: October 15 at 11:59 PM. Submit on Canvas.

Problem 1 (Relativistic stress tensor): In this problem we will explore $T^{\mu\nu}$ in relativistic field theory.

15 **A:** First, consider some generic theory whose Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ that is invariant under the Lorentz group (is relativistic). Using formulas from Lecture 14, argue that if $\mathcal{L}$ is invariant, then the stress tensor is symmetric: $T^{\mu\nu} = T^{\nu\mu}$. This is often stated as a more general consequence of Lorentz invariance, although you only need to derive it for this specific example.

15 **B:** In Lecture 15, we derived the effective field theory for electromagnetism. Use Noether's Theorem, as stated in Lecture 14, to derive the conserved stress tensor $T^{\mu\nu}$. You should find that your formula is not symmetric: $T^{\mu\nu} \neq T^{\nu\mu}$, in contrast with what was stated in **A**. But, argue that you can modify

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda J^{\lambda\mu\nu} \tag{1}$$

and that for some choice of $J^{\lambda\mu\nu}$, after using the equations of motion, you can shift $T^{\mu\nu}$ to

$$T^{\mu\nu} = F^{\mu\alpha}\eta_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{2}$$

This is the more common form for $T^{\mu\nu}$ in relativistic electromagnetism.

15 **C:** Under infinitesimal Lorentz transformations, in the theory of relativistic electromagnetism, write how A_μ transforms, if

$$x^\mu \to x^\mu + \epsilon^\mu{}_\nu x^\nu. \tag{3}$$

You may want to review the theory of Lorentz transformations from Lecture 6. Then, use results from Lecture 14 to deduce the conserved Noether current associated to an infinitesimal Lorentz transformation: $J^\mu{}_\alpha{}^\beta \epsilon^\alpha{}_\beta$. Relate this procedure to your answer from **B**.

Problem 2 (Wrong effective field theory of a solid): Suppose that we had instead tried to build the Lagrangian for an elastic solid based on the following principles: (*1*) the slow degrees of freedom are the displacement fields ϕ_i; (*2*) we have the shift symmetry $\phi_i \to \phi_i + c_i$ for any constant c_i; (*3*) we have spacetime translation symmetry; (*4*) we have combined rotational symmetry for the displacement ϕ_i and the coordinates x_i. (We are interested in studying the isotropic solid.)

20 **A:** If these are all of the symmetries, argue that the most general effective field theory with $\mathcal{L}$ (not only the equations of motion) invariant under all transformations is

$$\mathcal{L} = \frac{1}{2}\rho\partial_t\phi_i\partial_t\phi_i - \frac{1}{2}A_1\partial_i\phi_j\partial_i\phi_j - \frac{1}{2}A_2\partial_i\phi_i\partial_j\phi_j + \cdots . \tag{4}$$

Argue that the only stability constraints are that (if $\rho > 0$) $A_1 + A_2 \geq 0$ and $A_2 \geq 0$.

10 **B:** Identify an unambiguous and physically testable prediction that *disagrees* between this EFT, and the one from Lectures 17-19.[1] This EFT is wrong, and the one from Lectures 17-19 was correct. Explain why this one is wrong, and what physical demand we missed when building (4).

[1] *Hint:* Think about the normal modes carefully.

25 **Problem 3 (Piezoelectricity):** In certain crystal lattices with broken inversion and time-reversal symmetry, it is possible to find piezoelectricity – the crystal exhibits strain in response to an electric field. In this problem, we will consider this explicitly, assuming for simplicity that the electric field E_i is a fixed external parameter in the EFT of a solid:

$$\mathcal{L} = \mathcal{L}_0 + E_i \alpha^{IJK} \partial_i \sigma^I \partial_j \sigma^J \partial_j \sigma^K \tag{5}$$

where $\mathcal{L}_0$ is the solid Lagrangian from Lecture 17.[2]

Follow Lecture 18 and calculate the stress tensor of the solid in the presence of this piezoelectric term. Then expand around equilibrium – $\sigma^I = \delta^I_i (x_i - \phi_i)$. Assuming that the crystal is under no net stress, determine the equilibrium configuration (in particular, strain) of an elastic solid with three-fold rotation symmetry in the 12-plane, for which[3]

$$\alpha^{112} = \alpha^{121} = \alpha^{211} = -\alpha^{222} = \alpha_0, \tag{6}$$

in the presence of an infinitesimally small E_i. For simplicity in your answer, you should take $\mathcal{L}_0$ to be the isotropic solid Lagrangian, even though there are more terms that you could have added to $\mathcal{L}_0$ for a solid with this reduced symmetry pattern.

Problem 4 (Inflation): In this problem we will build a toy effective field theory for inflation – the rapid expansion of the early universe. To avoid a full (general relativistic) description of gravity, we assume that the only gravitational degree of freedom is the "scale factor" a. For the purposes of this problem, the existence of a requires that the action S be invariant under the following transformations:

$$x_i \to \lambda x_i, \tag{7a}$$

$$a \to \lambda a \tag{7b}$$

for constant $\lambda \neq 0$. Here x_i denote spatial coordinates only. You should also assume spacetime translation invariance, and spatial rotation invariance. However, do not assume Lorentz boost invariance, or time-reversal symmetry.

5 **A:** Inflation is driven by the presence of matter in the universe, which we will capture by the existence of another field ϕ. Show that the most general effective field theory, keeping lowest non-trivial orders in derivatives, has Lagrangian

$$\mathcal{L} = -A(a,\phi) - \frac{1}{2}B(a,\phi)\partial_i\phi\partial_i\phi + C(a,\phi)\partial_t\phi + \cdots . \tag{8}$$

What are the constraints on A, B and C? Assume there are 3 space and 1 time dimension.

5 **B:** Look for solutions to the Euler-Lagrange equations where a and ϕ only depend on t. Argue that, in a specific limit (which you should figure out, and may require "fine-tuning" of A or C), you may approximate that $\phi = \phi_0$ is a constant, while

$$a(t) \approx a_0 \mathrm{e}^{Ht} \tag{9}$$

where a_0 and H are constants which (if appropriate) you should relate to the parameters of your EFT.

10 **C:** Plug in the equilibrium solution (9) into (8), and obtain $\mathcal{L}$ that depends on ϕ alone. Expand $\mathcal{L}$ to quadratic order in fluctuations around the background from **B** Describe, as best you can, what "normal modes" look like in this theory, adding subleading terms to $\mathcal{L}$ if necessary to get a sensible theory.

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[2] Notice that E_i has a "space frame" index, because the electric field does not have to be oriented in the same way as the crystal!

[3] Notice that $\alpha^{IIJ} = 0$ and α^{IJK} is fully symmetric among its indices.