PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 7

Due: October 22 at 11:59 PM. Submit on Canvas.

Problem 1 (Time-dependent Hamiltonian): Consider the following Lagrangian, describing a particle moving on the line (configuration space $\mathbb{R}$), with a time-dependent

$$L = \frac{1}{2}m\dot{x}^2 - F(t)x - g(x)\dot{x}. \tag{1}$$

15 **A:** Take $g(x) = 0$. Perform a Legendre transform, identifying the momentum p canonically conjugate to x. Identify the Hamiltonian $H(x, p, t)$ and evaluate Hamilton's equations, confirming that they are equivalent to the Euler-Lagrange equations for the Lagrangian system.

10 **B:** Explain why $\{H, H\} = 0$. Does this imply that H is a conserved quantity? Why or why not?

15 **C:** Argue that $g(x)$ should not affect the physics. Does it affect the Hamiltonian, however? If it does, are Hamilton's equations modified? If your answer to the last question is yes, reconcile with the statement that $g(x)$ was an unphysical term in L.

Problem 2 (Multipolar conservation laws): Consider a Hamiltonian system with phase space $\mathbb{R}^{2N}$, describing N particles moving in one spatial dimension, with phase space coordinates x_a and p_a obeying $\{x_a, p_b\} = \delta_{ab}$ ($a = 1, \ldots, N$ labels each particle). In this problem we will consider theories that conserve the following quantities: for non-negative integer n,

$$Q_n = \sum_{a=1}^{N} x_a^n. \tag{2}$$

We can intuitively interpret Q_n as the n^{th} multipole moment of the mass distribution of a collection of n equal mass particles.

15 **A:** Consider a theory in which the Hamiltonian takes the form

$$H = \sum_{a=1}^{N-1} f(x_a, x_{a+1}, p_a, p_{a+1}). \tag{3}$$

This H could describe a chain of particles with nearest-neighbor interactions, for example. Suppose that $\{H, Q_k\} = 0$, where $k > 0$. What is the most general form of f?

15 **B:** Now suppose that we want a theory for which $\{H, Q_k\} = \{H, P\} = 0$, where P is the total momentum

$$P = \sum_{a=1}^{N} p_a. \tag{4}$$

Find the algebra formed by the conserved quantities, and show that it can be written in terms of the Q_ns and P alone. Your answer is called a **multipole algebra**.

30 **Problem 3 (Finding the full symmetry algebra):** Consider a Hamiltonian theory on phase space $\mathbb{R}^6$, with standard Poisson brackets $\{x_i, p_j\} = \delta_{ij}$. Suppose that we are given a Hamiltonian H which is invariant under a symmetry algebra which contains the following 2 generators: $L_x + \alpha p_x$ and $L_y + \alpha p_y$. Here α is some constant, with dimensions of length, if you prefer; $L_i = \epsilon_{ijk} x_k p_k$ denotes angular momentum.

By evaluating Poisson brackets of generators, find the full symmetry algebra (which has more than 2 generators). What is the most general H invariant under this full symmetry?

15 **Problem 4 (Non-reciprocal solid):** There is a lot of recent interest in "active matter", the study of dynamical systems whose constituent particles are "driven by batteries." One dramatic demonstration of active matter arises in a microscopic model of an "odd elastic solid", in which the constituent particles have interactions that violate Newton's Third Law (two particles act with equal and opposite forces).

An explicit example of such an odd elastic solid consists of interacting particles arranged in a two-dimensional square lattice. As depicted in Figure 1, every alternating particle is colored with either red or blue, and for each pair of adjacent particles, there is an "anti-reciprocal" force $\mathbf{F}$ that acts perpendicular to the distance between them. The magnitude of this force $|\mathbf{F}|$ depends only on the distance between each pair of particles. From a Newtonian perspective, the active equations of motion for a particle a (either red or blue) in the lattice then take the form

$$m\ddot{\mathbf{x}}_a + \gamma\dot{\mathbf{x}}_a = \sum_{b\sim a} \mathbf{F}_{ab}, \tag{5}$$

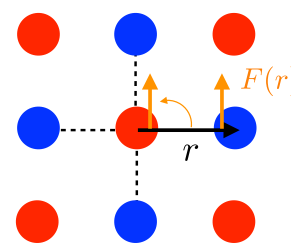

Figure 1: A two-dimensional square lattice of interacting red/blue particles. The direction of the force between red/blue particles is always rotated by 90° from the displacement from red to blue, and has magnitude $F(r)$ for all particle pairs.

where $\mathbf{F}_{ab}$ is the "anti-reciprocal" force between a and b, and $a \sim b$ denotes nearest neighbors in the square lattice. For simplicity in what follows, consider the highly overdamped limit in which $m \to 0$, such that you can approximate the equations of motion (5) as first order.

Show that if $m \to 0$, this active system is a Hamiltonian dynamical system; namely, identify a suitable phase space, Poisson bracket and Hamiltonian function that reproduces (5).