PHYS 5210: Graduate Classical Mechanics **Fall 2024**

Homework 8

Due: October 29 at 11:59 PM. Submit on Canvas.

Problem 1 (Symplectic reduction from 4 to 2 dimensions): Consider the phase space $\mathbb{R}^4$ with canonical coordinates (x_1, x_2, p_1, p_2). In this problem, we will build two-dimensional symplectic manifolds via symplectic reduction (Lecture 23).

15 **A:** Let's start with a physical illustration of the idea. Consider the diatomic molecule Lagrangian

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2 \tag{1}$$

from Lecture 5. Perform the Legendre transform and show that you obtain Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}k(x_1 - x_2)^2. \tag{2}$$

Show that $\{p_1 + p_2, H\} = 0$, and explain what this physically means.

15 **B:** Show that performing symplectic reduction on $\mathbb{R}^4$ using the canonical transformations generated by

$$F_1 = p_1 + p_2 \tag{3}$$

gives you a new phase space $\mathbb{R}^2$. What are a natural set of canonical coordinates on the reduced phase space? What is their physical interpretation, in the context of the diatomic molecule?

15 **C:** As an unusual modification of the above, consider instead performing symplectic reduction on $\mathbb{R}^4$ by the function

$$F_2 = p_1^2 + p_2^2. \tag{4}$$

Show that the symplectic manifold that results is a cylinder.

15 **D:** Next do symplectic reduction by

$$F_3 = x_1^2 + x_2^2 + p_1^2 + p_2^2. \tag{5}$$

Show that the symplectic manifold M that comes from performing symplectic reduction is the two-sphere S^2, which is therefore a symplectic manifold.[1]

[1] *Hint:* First show that the level sets of F_3 are the three-dimensional sphere S^3, and that the canonical transformations generated by F_3 are periodic in time. Can you try to match each point on the symplectically reduced phase space to a point (x_1, p_1)? What goes wrong?

Problem 2 (Rotation symmetry with an unusual symplectic form): Consider the following action describing a Hamiltonian system (in the generalized sense of Lecture 24) on phase space $\mathbb{R}^4$: for some positive function $A > 0$,

$$S = \int \mathrm{d}t \left[A\left(\sqrt{x^2+y^2}\right)(p_x\dot{x} + p_y\dot{y}) - H(x,y,p_x,p_y) \right]. \tag{6}$$

15 **A:** Identify the symplectic potential λ_α and then calculate the symplectic form $\omega_{\alpha\beta} = \partial_\alpha\lambda_\beta - \partial_\beta\lambda_\alpha$. What are Hamilton's equations, written in terms of the symplectic form?

10 **B:** Explain why the following infinitesimal transformation is canonical:

$$x \to x - \epsilon y, \tag{7a}$$
$$y \to y + \epsilon x, \tag{7b}$$
$$p_x \to p_x - \epsilon p_y, \tag{7c}$$
$$p_y \to p_y + \epsilon p_x. \tag{7d}$$

Find the function on phase space – call it angular momentum M – that generates this canonical transformation.

15 **C:** Show that a general Hamiltonian of the form

$$H = \frac{p_x^2 + p_y^2}{2m} + V\left(\sqrt{x^2+y^2}\right) \tag{8}$$

is symmetric under the canonical transformation generated by M. Replace (x, y) with polar coordinates (r, θ), and show that performing symplectic reduction via (7) leads to an effective Hamiltonian

$$H_{\mathrm{eff}}(r, p_r) = \frac{1}{2mA(r)^2}\left[p_r^2 + \frac{M^2}{r^2}\right] + V(r) \tag{9}$$

where we have *defined* p_r such that $\{r, p_r\} = 1$.

Problem 3 (Symplectic reduction for rigid body rotation): In this problem we will study how various rigid body rotation problems, in the presence of certain symmetries, lead to interesting symplectic reductions from the starting phase space $\mathrm{T}^*\mathrm{SO}(3)$.

10 **A:** Consider a fully asymmetric spinning top, with $I_1 \neq I_2 \neq I_3$. As in Lecture 11, suppose that the top is placed in a uniform gravitational field. Assume that the center of mass of the spinning top is aligned with the 3-axis. Argue that the resulting Lagrangian has one obvious continuous symmetry generator.[2] Although you don't need to explicitly derive the Hamiltonian, perform symplectic reduction on the phase space of the spinning top and describe what the resulting phase space is.

10 **B:** Now, suppose that the asymmetric top rotates in the absence of any external forces. Using the Hamiltonian formalism of Lecture 25, identify the generators of canonical transformations that leave the remaining Hamiltonian invariant, and explain their physical interpretation. Perform symplectic reduction under *all* such symmetry generators and identify the resulting symplectic manifold.[3]

[2] *Hint:* use Euler angles.
[3] *Hint:* for $R_{iI} \in \mathrm{SO}(3)$, $\epsilon_{ijk} = \epsilon_{IJK}R_{iI}R_{jJ}R_{kK}$.