Homework 9

Due: November 5 at 11:59 PM. Submit on Canvas.

Problem 1 (Chiral anomaly): On Homework 5, we discussed the chiral anomaly in 1+1 dimensions. In this problem, we revisit such a system from a Hamiltonian field theory perspective instead. Let $\rho(x)$ denote the density of a conserved charge, and consider Hamiltonian

$$H[\rho] = \int \mathrm{d}x \epsilon(\rho(x)). \tag{1}$$

In the presence of a chiral anomaly, the Poisson bracket[1]

$$\{\rho(x), \rho(y)\} = -\partial_x \delta(x-y). \tag{2}$$

Assume that integrals over the coordinate x have vanishing boundary terms at $\pm\infty$.

20 **A:** By calculating $\{\int \mathrm{d}x f(x)\rho(x), \int \mathrm{d}y g(y)\rho(y)\}$, confirm that the Poisson bracket defined by (2) is antisymmetric. Why will it also obey the Jacobi identity? Following Lecture 26, evaluate $\partial_t \rho = \{\rho, H\}$ and deduce the Hamiltonian equations of motion.

15 **B:** To compare with Homework 5, we would like to find a Lagrangian formulation of this problem. However, due to the derivative in (2), this is not immediately obvious. A convenient trick is to introduce additional "phase" degrees of freedom $\phi(x)$, with Poisson bracket $\{\phi(x), \phi(y)\} = 0$ and

$$\{\phi(x), \rho(y)\} = \delta(x-y). \tag{3}$$

In other words, phase and density are thus canonical conjugate degrees of freedom. Using this definition, conclude that the Hamiltonian field theory action takes the form of[2]

$$S[\rho, \phi] = \int \mathrm{d}x \left[\rho \partial_t \phi + \frac{1}{2}\partial_t \phi \partial_x \phi - \epsilon(\rho)\right] \tag{4}$$

Show that you can "integrate out" ρ, which now looks like a Lagrange multiplier,[3] and that after doing so, the resulting theory expressed only in terms of ϕ has the reparameterization symmetry from Homework 5.

[1] The function $\partial_x \delta(x)$ is defined such that $\int \mathrm{d}x f(x) \partial_x \delta(x) = -f'(0)$.

[2] *Hint:* Notice that the Poisson brackets are not field-dependent. If we were talking about a finite-dimensional phase space, we could then try to write $L = \frac{1}{2}\omega_{\alpha\beta}\xi^\alpha \dot{\xi}^\beta - H$. You can try to do the same thing in field theory. Lastly, you may use the following block matrix identity: $\begin{pmatrix} 0 & \delta \\ -\delta & D \end{pmatrix}^{-1} = \begin{pmatrix} D & -\delta \\ \delta & 0 \end{pmatrix}$.

[3] *Hint:* Think about transitioning from Hamiltonian mechanics back to Lagrangian mechanics!

Problem 2 (Parabolic coordinates): In this problem, we will study a seemingly complicated problem which can be cleverly solved using the Hamilton-Jacobi method. Consider a system in cylindrical coordinates (ρ, ϕ, z) with canonical conjugate momenta (p_ρ, p_ϕ, p_z), and Hamiltonian

$$H = \frac{1}{2m}\left[p_\rho^2 + \frac{p_\theta^2}{\rho^2} + p_z^2\right] - \frac{k}{\sqrt{\rho^2 + z^2}} - Fz. \tag{5}$$

This models the electron orbiting the proton in a hydrogen atom, placed in a uniform electric field.

It is far from obvious (but we will verify, true) that this problem can be solved by moving to parabolic coordinates (ξ, φ, η):

$$z = \frac{\xi - \eta}{2}, \tag{6a}$$

$$\phi = \varphi, \tag{6b}$$

$$\rho = \sqrt{\xi\eta}. \tag{6c}$$

20 **A:** Find expressions for (ξ, η, φ) in terms of (ρ, ϕ, z). Then use a type-2 generating function of the form $F(\rho, \phi, z, P_\eta, P_\varphi, P_\xi)$ to find expressions for the new canonical momenta $(P_\eta, P_\varphi, P_\xi)$ in terms of the old (p_ρ, p_ϕ, p_z).[4] Lastly, re-write H in terms of the new coordinates $(\eta, \varphi, \xi, P_\eta, P_\varphi, P_\xi)$.

20 **B:** Show that you can solve the Hamilton-Jacobi equation, up to quadratures, using separation of variables, as in Lecture 28.[5] You do not need to worry about finding explicit formulas for the trajectories in terms of the solution to the Hamilton-Jacobi equation.

Problem 3 (Transient forcing): Consider a non-relativistic particle of mass m in the presence of a time-dependent potential, described by the Hamiltonian

$$H = \frac{p^2}{2m} + V(x - ut), \tag{7}$$

where $u > 0$ is a constant.

15 **A:** Perform a type-2 canonical transformation (as simple as possible!) to a new coordinate system in which $X = x - ut$ is the new position coordinate. What is the new momentum coordinate P? Show that the new Hamiltonian

$$H' = \frac{P^2}{2m} - uP + V(X) \tag{8}$$

is now time-independent.

10 **B:** Describe the trajectory of the particle qualitatively, assuming that – in the initial coordinates – as $t \to -\infty$, the particle starts at $x = p = 0$. Assume that $V(X)$ vanishes for $|X| \geq a$. How independent of the details of $V(x)$ is your answer?

[4] *Hint:* What should $\partial F/\partial P_\eta$ be equal to, e.g.?

[5] *Hint:* You may need to multiply the Hamilton-Jacobi equation by a function of ξ and η to make it look separable.

Problem 4 (Symmetries, revisited): Consider phase space $\mathbb{R}^{2n}$ with canonical coordinates (x_i, p_i). In Lecture 23 we defined a continuous symmetry in Hamiltonian mechanics as an infinitesimal canonical transformation generated by a function F which obeys $\{F(x_i, p_i), H(x_i, p_i)\} = 0$.

10 **A:** Revisit that argument, and point out that there exist systems which – in Lagrangian mechanics – we would have called symmetric under the infinitesimal canonical transformation generated by $F(x_i, p_i)$, but which we missed in Lecture 23, as $\{F, H\} \neq 0$. Point out a simple concrete example of a system where this happens.[6] Under our original definition of symmetry in Lagrangian mechanics, what is the new condition that F and H need to obey in order for F to generate a symmetry?

10 **B:** As an alternate perspective, show that Noether's Theorem from Lecture 23 can be correctly restated as follows: if F generates a symmetry, then you can always find a "related" canonical transformation to (X_i, P_i), such that the new Hamiltonian H' obeys $\{F, H'\} = 0$. In your answer, explain why the construction is not pedantic, but "physical" – in particular, give a sensible physical interpretation to the transformation to new coordinates (X_i, P_i).

[6] *Hint:* Think about all of the examples that we have seen so far in the class.