PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 10

Euler's equations

September 18

Config space of rigid body rotation: $SO(3)$

$\{3\times 3$ matrices R w/ $R^T R = 1,$ $\det(R) = 1\}$.

R_{iI} = rotation from body frame (I) $\rightarrow$ space frame (i)

$$R_{iI}R_{jI} = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}, \qquad R_{iI}R_{iJ} = \delta_{IJ}$$
$$(RR^T = 1) \qquad\qquad (R^T R = 1)$$

Find a Lagrangian?

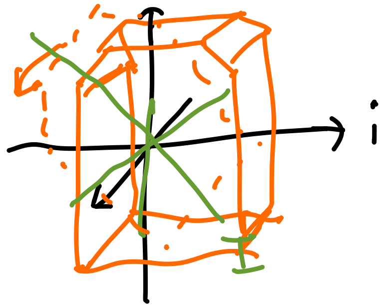
$$L = L_0(R_{iI}, \dot{R}_{iI}) + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$$

6 ind.

Lagrange multiplier: $\Lambda_{IJ} = \Lambda_{JI}$

Interested in an effective theory of slow dynamics

Symmetries of rigid body rotation?



Body frame symmetry?
↳ could be none.

Expect: all choices of space frame coords ⇒ same physics.

rotating space frame ⇐ universe's rotational symmetry

$$S[R_{iI}(t)] = S[Q_{ij}R_{jI}(t)]$$

constant matrix in SO(3)

analogue of translation symmetry on SO(3).
[left-SO(3) symmetry]

no body frame: $S[R_{iI}] \neq S[R_{iJ}Q_{JI}]$ in general.

Symmetries to assume:

- left-SO(3)
- time-reversal ($t \to -t$)
- time-translation $\partial L/\partial t = 0$.

So: $$L = L_0(R_{iI}, \dot{R}_{iI}) + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$$

×2

Invariants:

$R_{iI}R_{iJ} = \delta_{IJ}$ $\qquad \underbrace{R_{iI}\dot{R}_{iJ}}_{\Omega_{IJ}}$ $\qquad \dot{R}_{iI}\dot{R}_{iJ}$

(return later) $\qquad$ ×2 in L

left SO(3): $(Q_{ii'}R_{i'I})(Q_{ij'}R_{j'J})$

but $Q_{ii'}Q_{ij'} = \delta_{i'j'}$ ...

Analogy to relativity: contract spacetime for Lorentz invariance...

- Are there any other invariants of SO(3)? (yes: $\varepsilon_{ijk}\dots$)
(ignore...)

Claim: $L = \frac{1}{2}\dot{R}_{iI}\dot{R}_{iJ}K_{IJ} + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ})$

arbitrary, symmetric: $K_{IJ} = K_{JI}$

EOMs: $$\frac{\delta S}{\delta \Lambda_{IJ}} = 0 = R_{iI}R_{iJ} - \delta_{IJ}$$

AND: $$\frac{d}{dt}(R_{iI}R_{iJ}) = 0 = \dot{R}_{iI}R_{iJ} + R_{iI}\dot{R}_{iJ}$$

↳ $\Omega_{JI} + \Omega_{IJ}$

$\Rightarrow \Omega = -\Omega^T$ (Ω = antisymmetric)

$\dot{R}_{iI} = R_{iJ}\Omega_{JI}$, because: $\underbrace{R_{iK}\dot{R}_{iI}}_{\Omega_{KI}} = \underbrace{R_{iK}R_{iJ}}_{\delta_{KJ}}\Omega_{JI}$

Interpretation: Ω = angular velocity around body frame axes. (instantaneously)

Λ symmetric, K sym

$$\frac{\delta S}{\delta R_{iI}} = 2\Lambda_{IJ}R_{iJ} - \frac{d}{dt}\left(K_{IJ}\dot{R}_{iJ}\right) = 0. \quad \text{or}$$

$$2\Lambda_{IJ}R_{iJ} = K_{IJ}\ddot{R}_{iJ} \quad \text{or}$$

$\times R_{iK}$: $$2\Lambda_{IK} = R_{iK}K_{IJ}\ddot{R}_{iJ}$$

1) Λ = symmetric Lagrange multiplier, so

$$2(\Lambda_{IK} - \Lambda_{KI}) = 0 = R_{iK}K_{IJ}\ddot{R}_{iJ} - R_{iI}K_{KJ}\ddot{R}_{iJ}$$

2) Try plugging in $\dot{R}_{iI} = R_{iJ}\Omega_{JI}$ ...

and $\ddot{R}_{iI} = R_{iJ}\dot{\Omega}_{JI} + (R_{iK}\Omega_{KJ})\Omega_{JI}$.

So: $$R_{iK} K_{IJ} \ddot{R}_{iJ} = R_{iK} K_{IJ} R_{iL}(\dot{\Omega}_{LJ} + \Omega_{LM}\Omega_{MJ}) = K_{IJ}(\dot{\Omega}_{KJ} + \Omega_{KM}\Omega_{MJ})$$

$$\underbrace{[(\dot{\Omega} + \Omega^2)K]_{KI}}_{\text{EOM: must be antisymmetric}}$$

Claim: w/ a bit of algebra, using $K = K^T$ and $\Omega = -\Omega^T$:

define $L_{IJ} = -L_{JI} = K_{IK}\Omega_{KJ} + \Omega_{IK}K_{KJ}$ $\{L = K\Omega + \Omega K\}$

ang. mom.

Then: $\dot{L}_{IJ} = L_{IK}\Omega_{KJ} - \Omega_{IK}L_{KJ}$ $\{\dot{L} = [L, \Omega]\}$

$$L_{IJ} = \begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix} \qquad \Omega_{IJ} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Then EOM: Euler's equations:

$$\dot{L}_1 = L_2\omega_3 - \omega_2 L_3$$
$$\dot{L}_2 = L_3\omega_1 - \omega_3 L_1$$
$$\dot{L}_3 = L_1\omega_2 - \omega_1 L_2$$

And if: $L = K\Omega + \Omega K \ldots$

work in basis where

$$K = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

mom. of inertia

$I_1 = K_2 + K_3$ $\quad I_2 = K_3 + K_1$ $\quad I_3 = K_1 + K_2$:

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$