PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 11
Euler angles

September 20

Rigid body rotation: configuration space SO(3)
[3x3 orthogonal matrix R_{iI}]

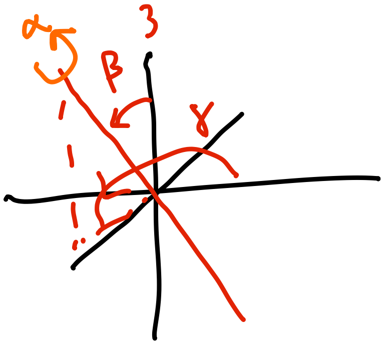
$$L = \frac{1}{2} K_{IJ} \dot{R}_{iI} \dot{R}_{iJ} + \Lambda_{IJ} \left(R_{iI} R_{iJ} - \delta_{IJ} \right)$$

symmetries transparent :) calculation difficult :(

find explicit coords for SO(3)?

Strategy 1: inspiration from QM. 3x3 rotation matrices for spin-1 particle?

$$R(t) = \exp\left[2\alpha \begin{pmatrix} 0 & -1 \cdot \cos\beta & \sin\beta \sin\gamma \\ 1 \cdot \cos\beta & 0 & -\sin\beta \cos\gamma \\ -\sin\beta \sin\gamma & \sin\beta \cos\gamma & 0 \end{pmatrix} \right]$$



In 3d → every rotation of this form
- pick axis "3"
- pick angle α to rotate by

Reveal geometry of $SO(3)$!

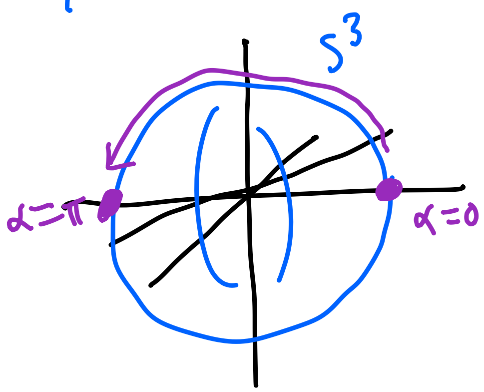
Evaluate $L = \frac{1}{2}(K_0 \delta_{IJ}) \dot{R}_{iI} \dot{R}_{iJ}$:

$$L = \# \Big[\underbrace{\dot{\alpha}^2 + \sin^2\alpha \, \dot{\beta}^2}_{\text{look like } S^2} + \sin^2\alpha \sin^2\beta \, \dot{\gamma}^2 \Big]$$

↳ look like S^3 (3-dim sphere)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

$x_1 = \sin\alpha \sin\beta \cos\gamma$ $\quad x_2 = \sin\alpha \sin\beta \sin\gamma$ $\quad x_3 = \sin\alpha \cos\beta$ $\quad x_4 = \cos\alpha$



So $SO(3) = S^3 / [\text{identify opposite points}]$

$= \{$ all lines passing thru origin of $\mathbb{R}^4 \} = \mathbb{RP}^3$

Strategy 2: Euler angles

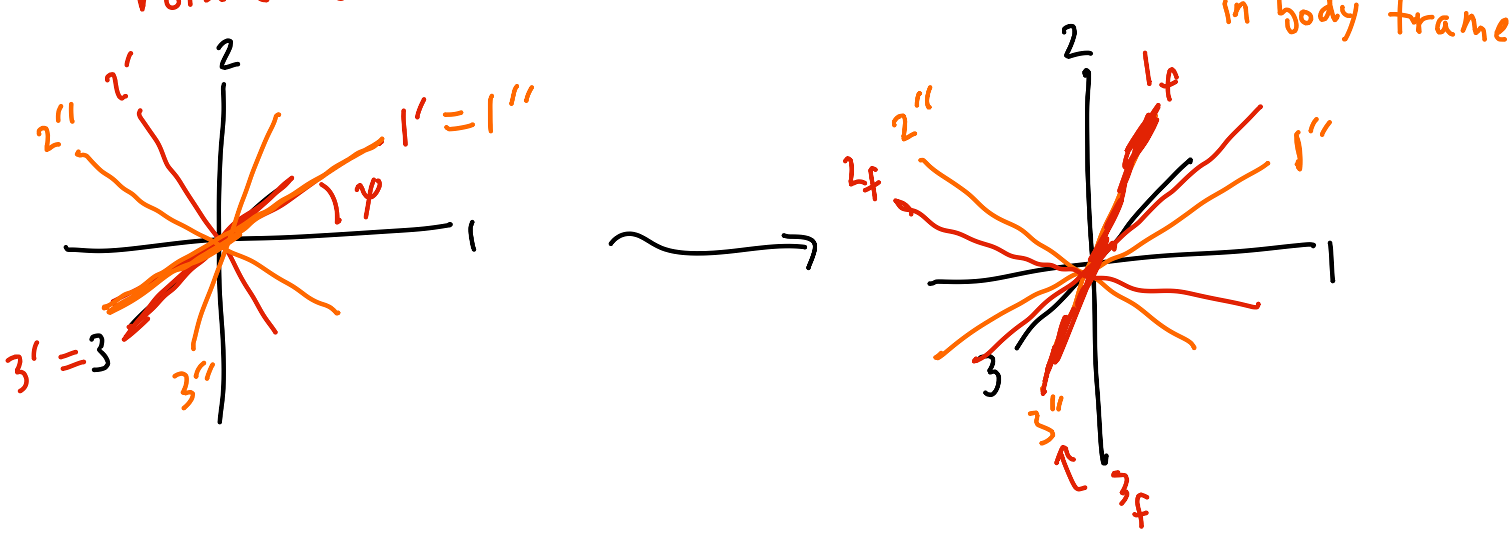
decompose $R = A(\phi) \, B(\theta) \, A(\psi)$

$$A(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotate around "3 axis"

$$B(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

rotate around 1 axis in body frame



Ugly formula for R:

$$R = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\theta\sin\phi \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$K_{IJ} = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}$$

(basis choice!)

AND

moment of inertia

$$I_{IJ} = \underbrace{K_{LL}}_{\mathrm{tr}(K)} \delta_{IJ} - K_{IJ}$$

$$= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$I_1 = K_2 + K_3$, etc...

$$L = \frac{1}{2} K_{IJ} \dot{R}_{iI}(\theta \dots) \dot{R}_{iJ}(\cdots) = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega^2 + \frac{1}{2} I_3 \omega_3^2$$

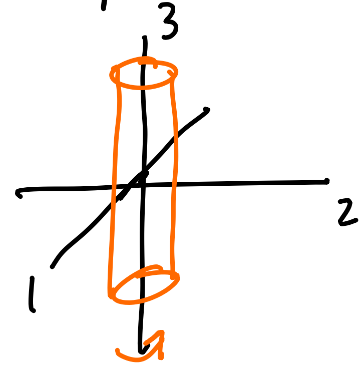
where
$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

2 simplifications that make Euler angles helpful:

1) $I_1 = I_2$ ("symmetric top")

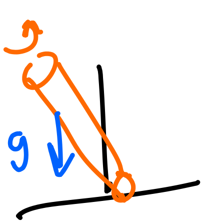
$\downarrow$

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) + \frac{I_3}{2}\left(\dot\phi \cos\theta + \dot\psi\right)^2$$



rotation sym in body frame around 3

2)



space frame: L can only depend on $\cos\theta$ in external potential?

Example: spinning top : (symmetric) $I_1 = I_2$ ✓

$$L = \frac{1}{2} I_1 (\sin^2\theta\, \dot\phi^2 + \dot\theta^2) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 - V(\cos\theta)$$

[most common: $V = mg\ell \cdot \cos\theta$]

Noether's Theorem! 3 constants of motion

① time-translation!

$$E = \frac{1}{2} I_1 (\sin^2\theta\, \dot\phi^2 + \dot\theta^2) + \frac{I_3}{2}(\dot\psi + \dot\phi\cos\theta)^2 + V(\cos\theta)$$

② ψ-translation: $\psi \to \psi + c$:

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3$$

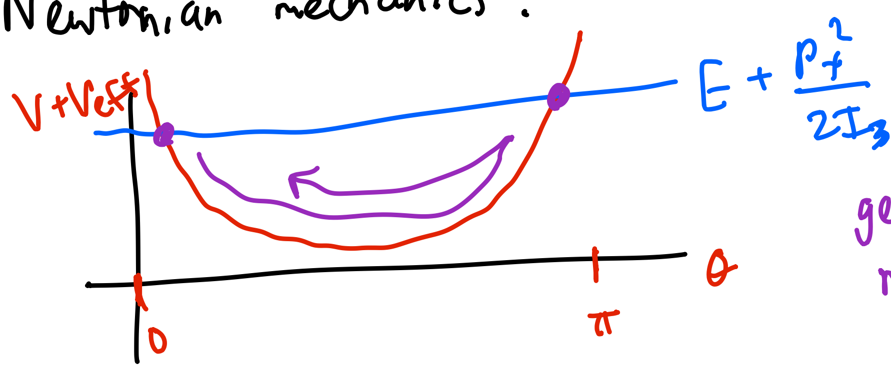
③ ϕ-translation: $\phi \to \phi + c$:

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + p_\psi \cos\theta = \text{const.}$$

Since Noether charges const. on phys. trajectories:

$$E = V(\cos\theta) + \frac{p_\psi^2}{2I_3} + \frac{I_1}{2}\dot\theta^2 + \underbrace{\frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}}_{V_{\text{eff}}}$$

Motion of "1d particle θ", analyze qualitatively w/ Newtonian mechanics:



generic oscillatory motion