PHYS 5210

Graduate Classical Mechanics

Fall 2024

Lecture 12

Lagrangian field theory

September 23

Field theory = systems w/ large ($N \gg 1$) DOF, w/ spatially local interactions & local in space fluctuations/dynamics important

Example (Lec 17–19): vibration of solid (phonons)

Let u_i denote displacement of i^{th} atom.

Lagrangian $L = \sum_{j=-\infty}^{\infty}\left[\frac{1}{2}m\dot{u}_j^2 - \frac{1}{2}k(u_j - u_{j-1})^2\right]$

$$\frac{\delta S}{\delta u_j} = 0 = \frac{\partial L}{\partial u_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}_j} = -k\left[(u_j - u_{j-1}) + (u_j - u_{j+1})\right] - m\ddot{u}_j$$

$$m\ddot{u}_j = -k\underbrace{\left[2u_j - u_{j+1} - u_{j-1}\right]}_{\text{spatial locality in EOM.}}$$

In example/real world... object (solid) ~ 1 m

but $-x_j + x_{j+1} \sim 10^{-10}$ m $\sim a + u_{j+1} - u_j$

Expect (verify!) that in slow dynamics... $\underbrace{u_j \simeq u_{j+1}}$.

Approximate $u_j \approx u(x=ja)$

Reasonable if "$\frac{du_j}{dj}$" $\ll u_j$

Then: $2u_j - u_{j-1} - u_{j+1} \simeq \underset{\text{eval. at } x=ja}{2u} - \left[u - a\partial_x u + \frac{a^2}{2}\partial_x^2 u + \cdots\right]$

$$- \left[u + a\partial_x u + \frac{a^2}{2}\partial_x^2 u + \cdots\right] \approx -a^2 \frac{\partial^2 u}{\partial x^2}$$

So if we approximate $u_j(t) \longrightarrow u(x=ja, t)$ $\quad \underline{[u(x,t)]}$ field

$$m\frac{\partial^2 u}{\partial t^2} = ka^2 \frac{\partial^2 u}{\partial x^2}.$$ If $\underset{\text{mass density}}{\rho} = \frac{m}{a}$ $\qquad K = ka$:

$\underline{\rho\, \partial_t^2 u = K\, \partial_x^2 u}$: wave equation! $\longrightarrow$ sound waves.

Goal: derive this equation from a Lagrangian

effective field theory (EFT)

Idea: Start w/ $$L = \sum_{j=-\infty}^{\infty} \left[\frac{m}{2}\dot{u}_j^2 - \frac{k}{2}\left[u_j - u_{j-1}\right]^2\right]$$

Express in terms of $u(x,t)$ instead of u_j.

$$L = \sum_{j=-\infty}^{\infty}\left[\frac{m}{2}\left(\frac{\partial u(x=ja,t)}{\partial t}\right)^2 - \frac{k}{2}\left[a\frac{\partial u(x=ja,t)}{\partial x} \cancel{- \frac{a^2}{2}\frac{\partial^2 u}{\partial x^2}} + \cdots\right]^2\right]$$

Neglect if $a\frac{\partial}{\partial x} \ll 1$;

sound wave wavelength $\gg a \sim 10^{-10}$ m.

$$\sum_j f(x=ja) \approx \int \frac{dx}{a} f(x) \quad , \qquad \text{So...}$$

Riemann

$$L = \int dx\, \mathcal{L} \qquad \text{where } \mathcal{L} = \frac{m}{2a}(\partial_t u)^2 - \frac{ka}{2}(\partial_x u)^2$$

"Lagrangian density" → "Lagrangian"

$$\mathcal{L} = \frac{\rho}{2}(\partial_t u)^2 - \frac{K}{2}(\partial_x u)^2$$

Expect principle of least action:

$S[u] = \int dx\, dt\, \mathcal{L}$ ... "physical trajectories" are $\frac{\delta S}{\delta u} = 0$.

Eventually: build $\mathcal{L}$ based on symmetries etc....

For today: evaluate $\frac{\delta S}{\delta u}$.

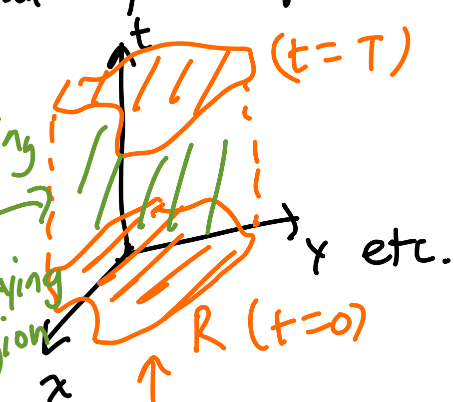
For ease of notation: group $(x, t, \cdots) = x^\mu$ no Lorentz invariance

Claim: as before, if $\mathcal{L}(u, \partial_\mu u)$ $\quad \frac{\partial u}{\partial x^\mu} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \cdots\right)$

S extremized if u is fixed on boundary of spacetime region

$$S[u] = \int_0^T dt \int_R dx \cdots \mathcal{L}(u, \partial_\mu u)$$

evaluating $\frac{\delta S}{\delta u}$ for u varying inside region



Define spacetime region to be Σ.

Also... $d^D x = dt\, dx \cdots$

$$= \prod_\mu dx^\mu$$

Claim (POLA): if $\bar{u}$ is "physical trajectory" (solves EOM),

$$\frac{d}{d\varepsilon} S[\bar{u} + \varepsilon \hat{u}]\Big|_{\varepsilon=0} = 0 \qquad (\text{if } \hat{u} = 0 \text{ on } \partial\Sigma)$$

$\partial\Sigma$ ← boundary of Σ.

$$\frac{dS}{d\varepsilon} = \int_\Sigma d^D x \; \frac{d\mathcal{L}(\bar{u}+\varepsilon\hat{u}, \partial_\mu \bar{u} + \varepsilon \partial_\mu \hat{u})}{d\varepsilon}\Big|_{\varepsilon=0}$$

$$= \int_\Sigma d^D x \left[\frac{\partial \mathcal{L}}{\partial u}\Big|_{\bar{u}} \hat{u} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}\Big|_{\bar{u}} \partial_\mu \hat{u} \right]$$

← Einstein $\sum_\mu$ implicit.

$$\frac{\partial \mathcal{L}}{\partial(\partial_t u)} \partial_t u + \frac{\partial \mathcal{L}}{\partial(\partial_x u)} \partial_x u + \cdots$$

$$= \int_\Sigma d^D x \; \hat{u} \underline{\left(\frac{\partial \mathcal{L}}{\partial u} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)} \right)}\Big|_{\bar{u}} + \oint_{\partial\Sigma} d^{D-1}x \; n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}\Big|_{\bar{u}} \hat{u}$$

n_μ: outward normal

But then: $\hat{u} = 0$ on $\partial\Sigma$ by assumption, so...

$$\frac{\delta S}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}$$

(generalize point particle: $\frac{\delta S}{\delta u} = \frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial \dot{u}}$)

Example (again): sound waves in solids:

$$\mathcal{L} = \frac{\rho}{2}(\partial_t u)^2 - \frac{K}{2}(\partial_x u)^2.$$

($\mathcal{L}$ has symmetry $u \to u+1$)

$$\frac{\delta S}{\delta u} = 0 = \cancel{\frac{\partial \mathcal{L}}{\partial u}} - \partial_t[\rho \partial_t u] - \partial_x[-K \partial_x u]$$

or $\rho \partial_t^2 u = K \partial_x^2 u$ ... same wave equation!