PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 13
Klein-Gordon equation

September 25

Lagrangian field theory:

$$S[\phi^a(x^\mu)] = \int d^D x \, \mathcal{L}(\phi^a, \partial_\mu \phi^a, \underbrace{\ldots}_{}, x^\mu)$$

$a = 1, \ldots, N$ space & time: $\mu = 1, \ldots, D$

Principle of least action → (field) Euler-Lagrange:

$$\frac{\delta S}{\delta \phi^a(x^\mu)} = 0 = \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} + \cdots$$

HW4 (higher derivatives)

Writing $\mathcal{L}$ in terms of symmetry invariant building blocks...:

spacetime translation: $x^\mu \to x^\mu + \varepsilon^\mu$ const.

$\hookrightarrow \dfrac{\partial \mathcal{L}}{\partial x^\mu} = 0 \qquad (\mathcal{L}(\phi^a, \partial_\mu \phi^a, \ldots))$

or: "shift symmetry": $\phi \to \phi + \varepsilon$ const.

$\hookrightarrow$ if $\mathcal{L}$ is invariant: $\mathcal{L}(\partial_\mu \phi)$

Suppose Lorentz invariance (relativistic / rotation & L. boost):

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \qquad \text{w/} \qquad \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$$

$$\text{“}\eta^{\mu\nu}\text{''} = \eta_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\phi(x) \to \phi(\Lambda \cdot x) \qquad \partial_\mu \phi \to \Lambda_\mu{}^\nu \partial_\nu \phi$$

Write down invariant building blocks? (Lorentz + spacetime translation)

$$\mathcal{L}(\phi, \underbrace{\partial_\mu\phi\,\partial^\mu\phi}_{\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi})$$

contract indices

$$\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -(\partial_t\phi)^2 + (\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2.$$

Simplest example field theory (relativistic):
take one field $\phi(x^\mu)$:

$$S[\phi] = \int d^Dx\, \mathcal{L}(\phi, \partial_\mu\phi\partial^\mu\phi) \qquad \text{(Taylor expand in } \partial_\mu)$$

$$= \int d^Dx \left[A(\phi) + B(\phi)\partial_\mu\phi\partial^\mu\phi + \cdots \right]$$

$$= \int d^Dx \left[\cancel{A_0} + A_1\phi + \frac{A_2}{2}\phi^2 + \cdots (B_0 + B_1\phi + \cdots)\partial_\mu\phi\partial^\mu\phi \right]$$

constant. doesn't affect EOM, so set $A_0 = 0$.

Aside: if $S[\phi] = \int d^Dx \left[\mathcal{L} + \partial_\mu K^\mu\right]$
$\to \int d^Dx\, \mathcal{L}$ (ignore total derivatives)

($\mathcal{L} \to -\mathcal{L}$ OK)

Now: $\mathcal{L} \to -A_1\phi - \frac{1}{2}A_2\phi^2 + \cdots - \frac{B_0}{2}\partial_\mu\phi\partial^\mu\phi$ (rescale ϕ B_0)

Euler-Lagrange equations:

$$\frac{\delta S}{\delta \phi} = 0 = \underbrace{-A_1 - A_2\phi}_{\frac{\partial \mathcal{L}}{\partial \phi}} - \partial_\mu(-B_0 \partial^\mu \phi)$$

$$- \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$\frac{\partial}{\partial(\partial_\mu\phi)}\left[-\frac{B_0}{2}\partial_\alpha\phi\partial^\alpha\phi\right]$$

$$= \frac{\partial}{\partial(\partial_\mu\phi)}\left[-\frac{B_0}{2}\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right] = -\frac{B_0}{2}\eta^{\alpha\beta}\left[\underbrace{\delta^\mu_\alpha}_{\frac{\partial\partial_\alpha\phi}{\partial\partial_\mu\phi}}\partial_\beta\phi + \partial_\alpha\phi\,\delta^\mu_\beta\right] = -B_0\partial^\mu\phi$$

Since $A_{1,2}, B_0$ constants... re-label $\tilde{\phi} = \phi + \frac{A_1}{A_2}$ $(\partial_\mu\tilde{\phi} = \partial_\mu\phi)$

$$0 = -A_2\tilde{\phi} + B_0\partial_\mu\partial^\mu\tilde{\phi}.$$

After change: $\tilde{\phi} = 0$ solves EOM ("physical trajectory")

$$S[\tilde{\phi}] = \int d^Dx\left[-\frac{B_0}{2}\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi} - \frac{1}{2}A_2\tilde{\phi}^2 + \cdots\right]$$

Rescale S by factor B_0: $m^2 = \frac{A_2}{B_0}$

$$S[\phi] = \int d^Dx\left[-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2\right]$$

(drop tilde)

Define: (dropping corrections) Klein-Gordon theory: relativistic spin-0 particle.

$i = x, y, z$

$$= \int d^Dx\left[\underbrace{\frac{1}{2}(\partial_t\phi)^2}_{\text{"kinetic"}} - \underbrace{\frac{1}{2}(\partial_i\phi)\partial_i\phi - \frac{1}{2}m^2\phi^2}_{\text{"potential"}}\right]$$

Usually after we deduce $\mathcal{L}$ based on symmetries... look for equilibrium... expand to quadratic order... calculate <u>normal modes</u>.

→ stability!

→ experimentally detectable...

For Klein-Gordon: $\phi = 0$ is equilibrium, $\mathcal{L}$ quadratic:

EOM: $\dfrac{\delta S}{\delta \phi} = 0$: $\quad m^2 \phi = \partial_\mu \partial^\mu \phi$

If EFT has spacetime translation... look for plane waves:

$$\phi \to \phi_0 e^{i k_\mu x^\mu} = \phi_0 e^{i(k_x x + k_y y + k_z z - \omega t)}$$

ϕ_0 ← const.

Plug-in: $m^2 \phi_0 = -k_\mu k^\mu \phi_0 \to m^2 = \omega^2 \overbrace{- k_x^2 - k_y^2 - k_z^2}^{k^2}$.

Dispersion relation: $\omega(\vec{k}) = \pm \sqrt{(m/\hbar)^2 + k^2}$

Restore speed of light $c \neq 1$: (*)

$$\hbar\omega = \pm\sqrt{(c^2 m)^2 + (c\hbar k)^2}$$

$$E = \pm\sqrt{(mc^2)^2 + (cp)^2}$$

So m in Klein-Gordon theory is interpreted as a particle mass...

Given dispersion relation, build general solution to $m^2\phi = \partial_\mu \partial^\mu \phi$:

$$\phi(x) = \underbrace{\int dk_x dk_y dk_z}_{} \sum_{\pm} e^{-i\omega_\pm(k)t + i\vec{k}\cdot\vec{x}} a_\pm(\vec{k})$$

(k_x, k_y, k_z)

only integrate over spatial k b/c ω = fixed by dispersion

$\omega = \pm\sqrt{k^2 + m^2}$

Why $m^2 > 0$: $\quad \omega = \sqrt{k^2 + m^2}$ is real for any k

if $m^2 < 0$: $\quad \omega = \sqrt{k^2 - m^2}$ and at $k = 0$:

"tachyon"

$\omega = \pm i m$

$\phi \sim e^{\pm m t}$ → instability.