PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 14
Noether's Theorem in field theory

September 27

Lagrangian field theory:

$$S[\phi^a(x^\mu)] = \int d^D x \; \mathcal{L}(\phi^a, \partial_\mu \phi^a, x^\mu)$$

$a = 1, \ldots, N$

$\mu = 1, \ldots, D$
spacetime coords

Pick S to be invariant under symmetries...

→ For point particles, if S invariant under cont. sym....

$\left(\frac{dQ}{dt} = 0 \text{ on phys. traj.}\right)$ Conserved quantity Q

Field generalization:

$\left(\partial_\mu J^\mu = 0 \text{ on phys. traj}\right)$ Conserved current J^μ

$= \frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} + \ldots$

(charge) density

Restrict to $D = 1$:

$0 = \frac{d}{dt} \text{"}\cancel{J^t}\text{"} \, Q$

Derivation of J^μ follows Lecture 3:

→ Define: continuous symmetry:

$x = \tilde{x} - \varepsilon X$

- $\phi^a \to \tilde{\phi}^a = \phi^a + \varepsilon \varphi^a + \cdots = \phi^a(\tilde{x}) + \varepsilon[\varphi^a(\tilde{x}) - X^\nu \partial_\nu \varphi^a(\tilde{x})]$
- $x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon X^\mu + \cdots$
- $\mathcal{L} \to \tilde{\mathcal{L}} = \mathcal{L} + \varepsilon \partial_\mu K^\mu$ (S invariant up to const.)

"Invariant building block/ Lagrangian" condition:

$$0 = \partial_\mu K^\mu + \mathcal{L}\partial_\mu X^\mu + X^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \varphi^a \frac{\partial \mathcal{L}}{\partial \phi^a} + (\partial_\nu \varphi^a - \partial_\nu X^\mu \partial_\mu \phi^a)\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi^a}$$

If satisfied, continuous symmetry;

Noether's Theorem: on sol'ns to EOM $\left(\frac{\delta S}{\delta \phi} = 0\right)$, $\partial_\mu J^\mu = 0$,

where $J^\mu = X^\mu \mathcal{L} + K^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}(\varphi^a - X^\nu \partial_\nu \phi^a)$

If we have conserved J^μ:

$$\underbrace{Q}_{\text{"Noether charge"}} = \int \underbrace{dx \cdots}_{\text{space only}} \underbrace{J^t}_{\text{"charge density"}}$$

$$\frac{dQ}{dt} = \int dx \frac{\partial J^t}{\partial t} = -\int dx \left[\frac{\partial J^x}{\partial x} + \cdots\right] = -\oint_{\text{at } \infty} dA\, \hat{n}\cdot\vec{J} = 0.$$

Example 1: Klein-Gordon theory $(m^2 = 0)$

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

symmetries: Lorentz-invariance $(X^\mu = \varepsilon^\mu{}_\nu x^\nu)$ cf. lec 6 (HW 6)

shift symmetry: $\phi \to \phi + \varepsilon$ ← const. $(\varphi = 1)$

space & time translation: $x^\mu \to x^\mu + \varepsilon^\mu$ $(X^\mu = \varepsilon^\mu)$

D of these ← ε^μ → const.

Noether current for shift symmetry? ($\varphi = 1$, $X = 0$, $K = 0$)

$$J^{\mu} = \cancel{\varphi}^{1} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} = -\partial^{\mu}\phi$$

raise lower

$$\partial_{\mu} J^{\mu} = 0 = -\partial_{\mu}\partial^{\mu}\phi \nearrow \text{equivalent to } \frac{\delta S}{\delta \phi} \text{ w/ } m^2 = 0.$$

Example 2: Spacetime translation symmetry (almost always have)

↳ D independent symmetries

① $x^1 \to x^1 + 1 \quad (X^1 = 1)$
$x^2 \to x^2$

or

② $x^2 \to x^2 + 1 \quad (X^2 = 1)$

Package together:

$$X^{\mu} = \varepsilon^{\mu} = \delta^{\mu}{}_{\nu'} (\varepsilon^{\nu'} \ldots)$$

identity

drop eventually ...

ν = "choice of translation"

So: $\varphi^a = 0$, $K^{\mu} = 0$ (assume), $X^{\mu} = \delta^{\mu}{}_{\nu'} \cancel{\varepsilon^{\nu'}}$ (ε implicit)

We expect: $J^{\mu}{}_{\nu'} \to J^{\mu}{}_{\nu'} \varepsilon^{\nu'}$

Given (as before): $\frac{\partial \mathcal{L}}{\partial x^{\mu}} = 0.$

Then: $$J^{\mu}{}_{\nu'} = \delta^{\mu}{}_{\nu'} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^a)} \partial_{\nu'}\phi$$

$\partial_{\mu} J^{\mu}{}_{\nu'} = 0$ for each ν'.

In relativistic theories:

$$J^{\mu}{}_{\nu'} \eta^{\nu'\rho} = T^{\mu\rho} :$$

energy-momentum / stress ⨯ current / tensor

(HW6: $T^{\mu\rho} = T^{\rho\mu}$ from Lorentz)

$$T^{\mu\rho} = \eta^{\mu\rho}\mathcal{L} - \partial^{\rho}\phi^a \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi^a)}$$

E.g. conservation of energy: $t \to t+1$

$T^{\mu t}$-component only: $E = \int d^3x \; T^{tt}$

(energy) density   time-translation

conservation of momentum: $x \to x+1$

$$P^x = \int d^3x \; T^{tx}$$

Example 3: $\mathcal{L} = i\bar{\psi}\partial_t\psi - \frac{1}{2m}\partial_x\bar{\psi}\partial_x\psi$ ← complex-valued: ($\bar{\psi}$ = complex conj of ψ.)

$$\frac{\delta S}{\delta\bar{\psi}} = 0 = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}$$

$$= i\partial_t\psi - \partial_x\left(-\frac{1}{2m}\partial_x\psi\right) \quad \text{or} \quad i\partial_t\psi = -\frac{1}{2m}\partial_x^2\psi$$

So this $\mathcal{L}$ → Schrödinger equation!
(e.g. superfluid – HW5)

Evaluate: $T^\mu{}_{t'} = \delta^\mu{}_{t'}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_t\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\partial_t\bar{\psi}$:

$$T^t{}_t = \cancel{i\bar{\psi}\partial_t\psi} - \frac{1}{2m}\partial_x\bar{\psi}\partial_x\psi - \cancel{i\bar{\psi}\delta^\mu{}_t\partial_t\psi}$$

$$T^x{}_t = -\frac{1}{2m}\partial_x\bar{\psi}\partial_t\psi - \frac{1}{2m}\partial_x\psi\partial_t\bar{\psi}$$

flip sign

energy density $= \frac{1}{2m}\partial_x\bar{\psi}\partial_x\psi$ ← $E = -\int dx\, \bar{\psi}\frac{\partial_x^2\psi}{2m}$

momentum density

$$T^t{}_x = -\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\partial_x\psi - \frac{\partial\mathcal{L}}{\partial(\partial_t\bar{\psi})}\partial_x\bar{\psi} = -i\bar{\psi}\partial_x\psi$$

Total momentum: $P_x = \int dx\ \bar{\psi}(-i\partial_x)\psi$

These expressions consistent w/ expectations from QM. :)