PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 15
Electromagnetism

September 30

E&M : physical fields $\vec{E}$ & $\vec{B}$

↳ expressed as $\underbrace{\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}}$

encode: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ① and $\nabla \cdot \vec{B} = 0$. ②

Lec 7: Define $A_\mu = (-\varphi, \vec{A})$, then

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \begin{array}{cc} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & B_x & 0 \end{pmatrix} & \begin{array}{c} t \\ x \\ y \\ z \end{array} \\ \begin{array}{cccc} t & x & y & z \end{array} & \end{array}$$

Goal: deduce $\mathcal{L}(A_\mu, \partial_\nu A_\mu, \ldots)$ that reproduces Maxwell's eqn's

Note: ① & ② come from

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$$

follows from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

③ $\nabla \cdot \vec{E} = \rho$

④ $\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J}$

w/ matter (!)

"Constraint:" A_μ not uniquely defined.

gauge transform: $A_\mu \to A_\mu + \partial_\mu \Lambda$ leave $F_{\mu\nu}$ invariant.

We'll take $\mathcal{L}$ invariant under gauge transform...

Simplest gauge-invariant building block: $F_{\mu\nu}$

enforce $\underbrace{\text{translation}}$ / Lorentz:

no x-dependence

$$\mathcal{L} = -\frac{1}{4} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{= F_{\mu\nu} F_{\alpha\beta} \eta^{\mu\alpha} \eta^{\nu\beta}} + \ldots$$

$\leftarrow \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

subleading in ∂_μ's...

leading order...

subleading: $(F_{\mu\nu} F^{\mu\nu})^2$ or $\partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu} + \ldots$

Reproduce Maxwell's equations:

$$\frac{\delta S}{\delta A_\alpha} = 0 = \cancel{\frac{\partial \mathcal{L}}{\partial A_\alpha}}^{\,0} - \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} = \frac{1}{4} \partial_\beta \frac{\partial}{\partial(\partial_\beta A_\alpha)} \left[F_{\mu\nu} F_{\rho\lambda} \eta^{\mu\rho} \eta^{\nu\lambda} \right]$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\beta A_\alpha)} = \frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_\beta A_\alpha)} = \delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu$$

$$\to 0 = \frac{1}{4} \partial_\beta \cdot 2 \left[(\delta^\beta_\mu \delta^\alpha_\nu - \delta^\beta_\nu \delta^\alpha_\mu) F_{\rho\lambda} \eta^{\mu\rho} \eta^{\nu\lambda} \right]$$

$$= \frac{1}{2} \partial_\beta \left[F^{\mu\nu} (\delta\text{'s}) \right] = \frac{1}{2} \partial_\beta \left[F^{\beta\alpha} - F^{\alpha\beta} \right] = \boxed{\partial_\beta F^{\beta\alpha} = 0}$$

Maxwell 3 & 4 (w/ $\rho, J = 0$)

③: $\alpha = t$: $\partial_\beta F^{\beta t} = \partial_i E^i = \nabla \cdot \vec{E} = 0$

$i, j \ldots$ space only

④ $\alpha = i$: $\partial_t F^{ti} + \partial_j F^{ji} \to \partial_t \vec{E} + \nabla \times \vec{B} = 0$

Normal modes for E&M? $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is quadratic in A

$$\partial_\beta F^{\beta\alpha} = \partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta) = 0 \text{ is linear in } A.$$

Spacetime translation symmetry: $A^\alpha = \underset{\text{const.}}{a^\alpha}\, e^{ik_\mu x^\mu} = a^\alpha e^{i(k_x x + \dots - \omega t)}$

$$k^\mu = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

rotate our coord axes

On plane waves: $\partial_\mu e^{ik_\nu x^\nu} = ik_\nu e^{ik_\nu x^\nu}$

So: $0 = k_\beta k^\beta A^\alpha - k_\beta k^\alpha A^\beta$

$$0 = (k^2 - \omega^2) A^\alpha - \begin{pmatrix} \omega \\ k \\ 0 \\ 0 \end{pmatrix} (-\omega A_t + k A_x) \qquad k^\mu = \begin{pmatrix} \omega \\ k \\ 0 \\ 0 \end{pmatrix}$$

Two solutions: $\alpha = y, z$: $A_t = A_x = 0$; $A_y = 1$ and $\underbrace{\boxed{\omega = \pm k}}_{\text{photons = normal modes}}$

dispersion relation $(\omega = ck)$ restore speed of light

What about $\alpha = t, x$?

$$(k^2 - \omega^2) A^t = \omega(kA_x - \omega A_t)$$
$$(k^2 - \omega^2) A^x = k(kA_x - \omega A_t)$$

$$\hookrightarrow \frac{A^t}{A^x} = \frac{\omega}{k}$$

Or: $A^t = i\omega \cdot \underset{\text{const.}}{a_0}$ $\qquad A^x = ik \cdot a_0$

or: $A^\mu = \partial^\mu(a_0 e^{ik_\nu x^\nu}) \longmapsto$ gauge transformation so this is unphysical

How to couple to charged particles?

$$S[A_\mu(x), x^\mu_A(\lambda)] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) + \underbrace{\sum_{\text{particle } A} \int d\lambda \left[-m\sqrt{\frac{dx^\mu_A}{d\lambda}\frac{dx_{A\mu}}{d\lambda}} + qA_\mu \frac{dx^\mu_A}{d\lambda}\right]}_{\text{lec 7}}$$

Note: $F_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} = 0$ b/c $(\text{antisym})_{\mu\nu}(\text{sym})^{\mu\nu} = 0$.

Suppose "heavy" particles, moving at non-relativistic speeds $\left(\left|\frac{dx_A}{dt}\right| \ll c\right)$

$$S[A_\mu(x), x^i_A(t)] = \dots + \sum_A \int dt \underbrace{\left[\frac{1}{2}m\sum_i \dot{x}^i_A \dot{x}^i_A - q A_t(\vec{x}_A) + q A_i(x_A)\dot{x}^i_A\right]}_{\text{E-L for } x_A:\quad m\ddot{x} = q(E + \dot{x}\times B)}$$

Vary w/r/t A_μ:

$$\frac{\delta S}{\delta A_t(x)} = 0 = \partial_\beta F^{\beta t} + \frac{\delta}{\delta A_t(x)} \sum_A \int dt \left[-q A_t(x^i_A, t)\right]$$

Define: $\dfrac{\delta A_t(x^\mu)}{\delta A_t(y^\mu)} = \delta^{(4)}(x-y)$

$$\rightarrow 0 = \partial_\beta F^{\beta t} - \sum_A q \int dt_0\, \delta^{(3)}(\vec{x}^i_A - \vec{x}^i)\,\delta(t-t_0)$$

$$= \nabla\cdot\vec{E} - \underbrace{\sum_A q\,\delta(\vec{x} - \vec{x}_A)}_{\rho(\vec{x}) = \text{charge density}}$$