PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 16
Coupling matter to electromagnetism

October 2

Goal: derive from our EFT:

$\nabla \cdot \vec{E} = \rho$ and $\nabla \times \vec{B} = \vec{J} + \partial_t \vec{E}$ $\quad \partial_\mu F^{\mu\nu} = J^\nu = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}$

Notice: $\partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_\nu J^\nu = 0$ b/c F antisym but $\partial\partial$ sym

Must couple to matter w/ $\underbrace{\text{conserved charge}}$...

Noether's Thm: matter theory has continuous symmetry?

Minimal example:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_2^2$$

Have continuous and $\underbrace{\text{spacetime-independent}}_{\text{"flavor"}}$ symmetry:

"flavor" $(x^\mu \to x^\mu)$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\lambda & +\sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

leaves $\mathcal{L}$ invariant.

(e.g. $\frac{m^2}{2}(\phi_1^2 + \phi_2^2)$ "length")

Only works for $\lambda = $ const. (More later...)

More elegant: use complex-valued field:

$$\phi = \phi_1 + i\phi_2 \quad \underline{\text{and}} \quad \bar{\phi} = \phi_1 - i\phi_2 .$$ (complex conjugates)

"Rotation symmetry" $\rightarrow$ U(1) symmetry: "$e^{i\theta}$"

$$\phi \rightarrow \phi e^{i\lambda} \qquad \bar{\phi} \rightarrow \bar{\phi} e^{-i\lambda}$$

$\phi\bar{\phi} = (\phi_1 + i\phi_2)(\phi_1 - i\phi_2) = \phi_1^2 + \phi_2^2$

Re-write

$$\mathcal{L} = -\cancel{\tfrac{1}{2}}\, \partial_\mu \phi\, \partial^\mu \bar{\phi} - \cancel{\tfrac{1}{2}} m^2 \boxed{\phi\bar{\phi}}$$

drop!

$$\frac{\delta S}{\delta \bar{\phi}} = \frac{\partial \mathcal{L}}{\partial \bar{\phi}} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\phi})} = -m^2 \phi - \partial_\alpha \partial^\alpha \phi$$ (Klein-Gordon eq.)

From U(1) symmetry, apply Noether's Thm:

for infinitesimal λ:

$$\phi \rightarrow \phi + \underbrace{i\phi}_{\varphi = i\phi} \cdot \lambda + \cdots \qquad \bar{\phi} \rightarrow \bar{\phi} \underbrace{-i\bar{\phi}}_{\bar{\varphi} = -i\bar{\phi}} \cdot \lambda + \cdots$$

From Lec 14:

$$J^\mu = \varphi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \bar{\varphi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})}$$
$$= i\phi\, \partial^\mu \bar{\phi} - i\bar{\phi}\, \partial^\mu \phi$$

Up to now: parameter $\underbrace{\lambda \text{ is constant}}$. ← independent of x

↳ global U(1) symmetry

But in E&M: gauge symmetry: $A_\mu \rightarrow A_\mu + \partial_\mu \underbrace{\lambda(x)}_{\text{gauge transform.}}$

Claim: to couple ϕ & $\bar{\phi}$ to A_μ, "promote global U(1)

↓

gauge U(1)"

What are invariant building blocks for $\mathcal{L}$ if:

$$\phi \to \phi e^{i\lambda(x)} \quad \& \quad \bar{\phi} \to \bar{\phi} e^{-i\lambda(x)}$$

e.g. $\phi\bar{\phi} \to \phi \cancel{e^{i\lambda(x)}} \bar{\phi} \cancel{e^{-i\lambda(x)}}$ invariant

as is $\partial_\mu(\phi\bar{\phi}) \cdot \partial^\mu(\phi\bar{\phi})$

If $\mathcal{L}(\phi\bar{\phi}, \partial_\mu(\phi\bar{\phi}), \ldots)$ then $\rho = \phi\bar{\phi}$ has $\mathcal{L}(\rho, \partial_\mu\rho, \ldots)$

To have meaningful EFT, couple to A_μ:

$\hookrightarrow A_\mu \to A_\mu + \partial_\mu \lambda$

Define "covariant derivative":

$$D_\mu\phi = \partial_\mu\phi - iA_\mu\phi$$

$$\partial_\mu(\phi e^{i\lambda}) - i(A_\mu + \partial_\mu\lambda)(\phi e^{i\lambda})$$

$$= e^{i\lambda}\left[\partial_\mu\phi + \cancel{i\phi\partial_\mu\lambda} - i(A_\mu + \cancel{\partial_\mu\lambda})\phi\right] = e^{i\lambda}D_\mu\phi$$

Similarly: $D_\mu\bar{\phi} = (\partial_\mu + iA_\mu)\bar{\phi}$ transforms as $e^{-i\lambda}D_\mu\bar{\phi}$

So our invariant $\mathcal{L}(\phi\bar{\phi}, D_\mu\phi D^\mu\bar{\phi}, F_{\mu\nu}F^{\mu\nu}, \ldots)$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Minimal EFT for $\underbrace{\text{charged matter + electromagnetism}}_{\text{U(1) gauge theory:}}$

$$\mathcal{L} = -D_\mu\phi D^\mu\bar{\phi} - m^2\phi\bar{\phi} - \cdots - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \cdots$$

(quantizing this + adding fermions $\to$ QED)

Check: do we indeed get Maxwell's equations out?

$$\frac{\delta S}{\delta A_\alpha} = \frac{\partial \mathcal{L}}{\partial A_\alpha} - \underbrace{\partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}}_{\text{lec 15: } +\partial_\beta F^{\beta\alpha}}$$

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = - \frac{\partial}{\partial A_\alpha}\left[(\partial_\mu \phi - iA_\mu \phi)(\partial^\mu \bar{\phi} + iA^\mu \bar{\phi})\right]$$

$$= -\left[-i\phi(\partial^\alpha \bar{\phi} + iA^\alpha \bar{\phi}) + i\bar{\phi}(\partial^\alpha - iA^\alpha \phi)\right]$$

$$= -\left[i\bar{\phi} D^\alpha \phi - i\phi \bar{D}^\alpha \bar{\phi}\right] = \underset{(+)}{-} J^\alpha$$

conserved U(1) current... but $\partial_\mu \to D_\mu$

definiton of conserved current in gauge field

So: $\partial_\beta F^{\beta\alpha} = J^\alpha$: Maxwell's equations.

Also calculate:

$$0 = \frac{\delta S}{\delta \bar{\phi}} = \frac{\partial \mathcal{L}}{\partial \bar{\phi}} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\phi})} = \frac{\partial}{\partial \bar{\phi}}\left[-m^2 \phi\bar{\phi} - (\partial_\mu - iA_\mu)\phi \cdot (\partial^\mu \bar{\phi} + iA^\mu \bar{\phi})\right] - \ldots$$

$$= -m^2 \phi - iA^\mu D_\mu \phi - \partial_\alpha(-D^\alpha \phi)$$

or $\quad 0 = -m^2 \phi + (\partial_\mu - iA_\mu) D^\mu \phi = -m^2 \phi + D_\mu D^\mu \phi$

generalization of Klein-Gordon eq. w/ $\partial_\mu \to D_\mu$.