PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 17
Effective field theory of solids

October 4

Goals for today:

1. what fields describe EFT of solid?
2. Symmetry?
3. Invariant building blocks
4. $\mathcal{L}$ that's stable

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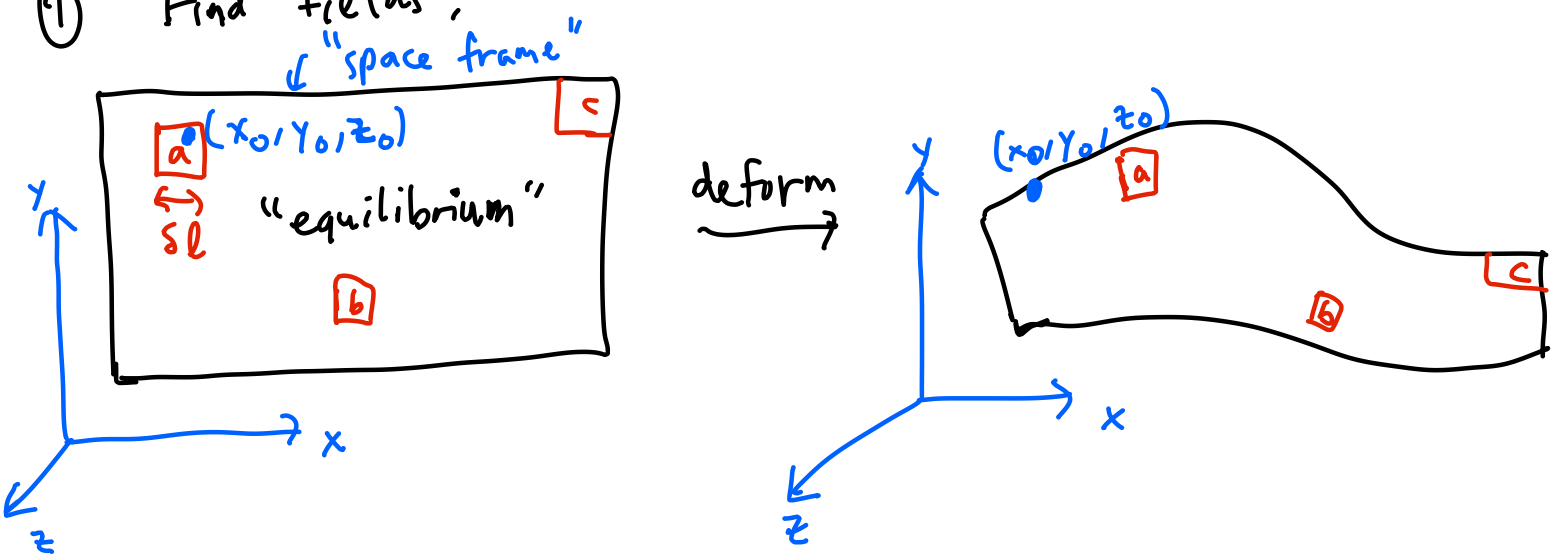
1. Find fields?



What "chunk" (a,b,c) is found at (x,y,z) in space frame?

chunk → σ (field)

$\sigma(\vec{x}, t)$ (Eulerian perspective)

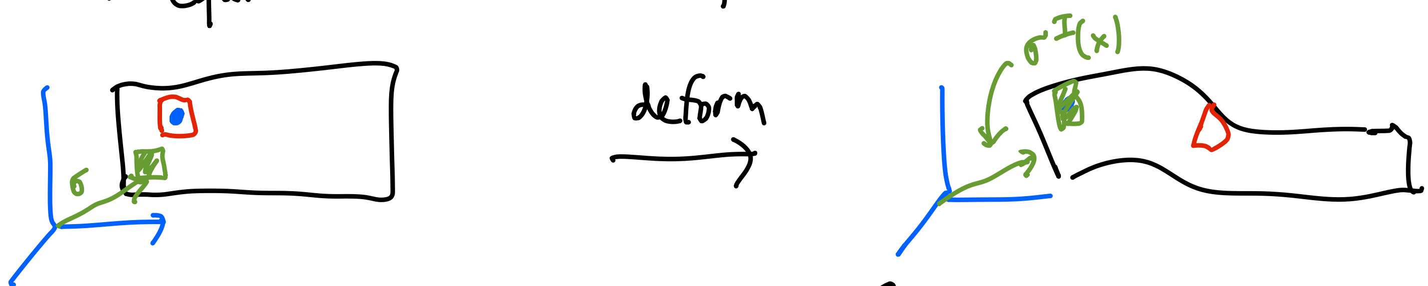
contrast: $x(\sigma, t)$ (Lagrangian persp.)

In equilibrium / at rest?

Chunks to specify: where piece is? at $\underbrace{\overset{\sigma^1}{[x_0, x_0+\delta\ell]} \times \overset{\sigma^2}{[y_0, y_0+\delta\ell]} \times \overset{\sigma^3}{[z_0, z_0+\delta\ell]}}_{\text{"}(\sigma^1,\ \sigma^2,\ \sigma^3)\text{"}}$

Analogy to rigid body notation: $\sigma^I(x,t)$ are fields.

in equilibrium: $\sigma^I = \delta^I_i x^i$ can be chosen.



② What are symmetries of EFT?

(space frame) space & time translation: $t \to t+\varepsilon$
$x_i \to x_i + \varepsilon_i$
($i j \cdots$ space (frame) only)

can also have time-reversal etc. $(t \to -t)$

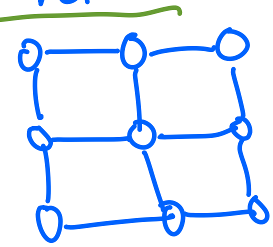
space frame rotation: $x_i \to Q_{ij} x_j$ for $Q \in SO(3)$

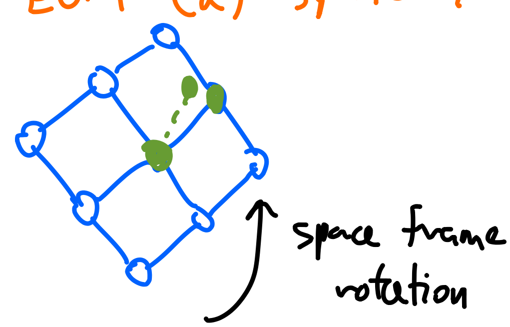
and: "body frame translation": $\sigma^I \to \sigma^I + \varepsilon^I$
(global motion of object is fine...)

↳ look at equilibrium: $\sigma^I = \delta^I_i x_i$
choice of eq changes if $\sigma^I \to \sigma^I + \varepsilon^I$
or $x_i \to x_i + \varepsilon_i$

(SSB) Spontaneous symmetry breaking: equilibria not invariant under EOM ($\mathcal{L}$) symmetry.

body frame rotation?





not in general... analogous to rigid body not having "right $SO(3)$" ... $\dot{R}_{iI} \dot{R}_{iJ} K_{IJ}$

in a fluid (liquid/gas): translation symmetry $\sigma \to \sigma + \varepsilon$ is not (fully) SSB

(3) Invariant building blocks? ($\mathcal{L}$ invariant)

- x_i & t-translation... $\mathcal{L}(\sigma^I, \partial_i \sigma^I, \partial_t \sigma^I)$
- σ^I-translation ... $\mathcal{L}(\partial_i \sigma^I, \partial_t \sigma^I)$
- x_i-rotation: $\mathcal{L}(\partial_i \sigma^I \partial_i \sigma^J, \partial_t \sigma^I)$

In class:

- isotropic solid (σ^I rotation): contract IJ at the end...

(4) What's $\mathcal{L}$? Need $\sigma^I = \delta^I_i x^i$ to be stable.

Finding invariants that vanish if $\sigma^I = \delta^I_i x_i$?

- $\partial_t \sigma^I \partial_t \sigma^I$ (time-rev & IJ indices contracted)
- $\partial_i \sigma^I \partial_i \sigma^J - (\ldots)_{eq} = \partial_i \sigma^I \partial_i \sigma^J - \partial_i(\delta^I_{i'} x_{i'}) \partial_i(\delta^J_{j'} x_{j'})$

$$= \partial_i \sigma^I \partial_i \sigma^J - \delta^I_i \delta^J_i$$

$$= \partial_i \sigma^I \partial_i \sigma^J - \delta^{IJ}$$

Now build $\mathcal{L}$: (Taylor expand)

$$\mathcal{L} = \underbrace{\frac{\rho}{2} \partial_t \sigma^I \partial_t \sigma^I}_{} - \frac{1}{8} \lambda^{\underbrace{IJ}\underbrace{KL}} (\partial_i \sigma^I \partial_i \sigma^J - \delta^{IJ})(\partial_j \sigma^K \partial_j \sigma^L - \delta^{KL})$$

ρ = mass density "kinetic" — "harmonic potential energy"

Stability: choose λ so it's positive:

$\lambda^{\boxed{IJ}\boxed{KL}}$ to be a positive-definite "matrix" on 9-dim space...

Impose body frame isotropy on λ:

"building λ out of tensors that are rotation-invariant":

δ^{IJ} is rotation invariant...

$$\lambda^{IJKL} = K\delta^{IJ}\delta^{KL} + \mu_1\delta^{IK}\delta^{JL} + \mu_2\delta^{IL}\delta^{JK}$$

<u>However:</u> $\lambda^{IJ\underline{KL}}$ is contracted into $\underbrace{\partial_j\sigma^K\partial_j\sigma^L - \delta^{KL}}_{\text{symmetric under } K\leftrightarrow L}$

So $\mathcal{L}$ only depends on "$\mu_1+\mu_2$":

$$\lambda^{IJKL} = K\delta^{IJ}\delta^{KL} + \tilde{\mu}\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}\right)$$

↑

"elastic moduli"/ Lamé coefficients...

for isotropic solid, microscopic details "absorbed" into ρ, K, $\tilde{\mu}$.

Theory of <u>elastic</u> solid: non-dissipative solid, deformed solid can return to eq.