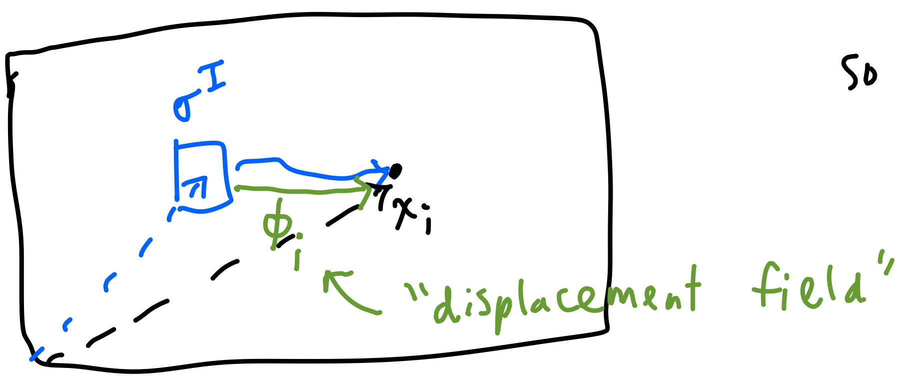
PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 18
Elastic stress tensor

October 7

Fields that enter EFT of solid:

$\sigma^I(x_i, t)$ = equilibrium location (σ^I) of chunk of solid sits at x_i at time t



so $I = 1, 2, 3$

$$\mathcal{L} = \frac{\rho}{2}\partial_t\sigma^I\partial_t\sigma^I - \frac{1}{8}\lambda^{IJKL}\left(\partial_i\sigma^I\partial_i\sigma^J - \delta^{IJ}\right)\left(\partial_j\sigma^K\partial_j\sigma^L - \delta^{KL}\right)$$

if solid is isotropic: $\lambda^{IJKL} = \tilde{K}\delta^{IJ}\delta^{KL} + \mu\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}\right)$

if not: λ^{IJKL} can have additional structure

Today: solids close to equilibrium:

$$\sigma^I = \delta^I_i\left(x_i - \phi_i\right) \rightsquigarrow \text{Taylor expand in } \phi$$

$$\partial_t \sigma^I \to -\partial_t \phi_i \delta_i^I$$

$$\partial_i \sigma^I \partial_i \sigma^J - \delta^{IJ} \longmapsto \partial_i \left[\delta_k^I (x_k - \phi_k) \right] \partial_i \left[\delta_\ell^J (x_\ell - \phi_\ell) \right] - \delta^{IJ}$$

$$= \delta_k^I \delta_\ell^J \left(\delta_{ik} - \partial_i \phi_k \right)\left(\delta_{i\ell} - \partial_i \phi_\ell \right) - \delta^{IJ}$$

$$= \underbrace{\delta_k^I \delta_\ell^J \delta_{k\ell}}_{\cancel{\delta^{IJ}}} - \delta_{ik} \partial_i \phi_\ell \delta_k^I \delta_\ell^J - \delta_{i\ell} \partial_i \phi_k \delta_k^I \delta_\ell^J - \cdots \cancel{-\delta^{IJ}}$$

$$\approx \text{"} -\partial^I \phi^J - \partial^J \phi^I \text{"} = -\delta_i^I \delta_j^J \underbrace{(\partial_i \phi_j + \partial_j \phi_i)}$$

strain tensor: $u_{ij} = \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i)$
$= u_{ji}$

$$\mathcal{L} = \frac{\rho}{2} \partial_t \phi_i \partial_t \phi_i - \frac{1}{8} \lambda_{ijk\ell} (\partial_i \phi_j + \partial_j \phi_i)(\partial_k \phi_\ell + \partial_\ell \phi_k) + \cdots$$

Symmetries?

- t & x_i translation symmetries
- if $\lambda_{ijk\ell} = \tilde{K} \delta_{ij} \delta_{k\ell} + \cdots$ have rotation symmetry
  $x_i \to Q_{ij} x_j$ and $\phi_i \to Q_{ij} \phi_j$ for orthogonal Q
- $\phi_i \to \phi_i + \varepsilon_i$ (shift)
- HW6: started out w/ 2 rotational symmetries in $\mathcal{L}(\sigma^I, \ldots)$?

From $\mathcal{L}$: normal modes? (Lec 19)
static property: (elastic) stress tensor

from Noether's Thm: x_i translation symmetry...
$\downarrow$
momentum conservation: spatial part of current T_{ij}?

Go back (for now) to $\mathcal{L}(\sigma, x)$

[reason: $\sigma^I = \delta^I_i(x_i - \phi_i)$ mixes $\phi \to \phi + \varepsilon$, $x \to x + \varepsilon$ symmetries]

(lec 14)

$$T_{ij} = \underbrace{\mathcal{L}}_{\cancel{\phi^2}} \delta_{ij} - \partial_j \sigma^I \frac{\partial \mathcal{L}}{\partial(\partial_i \sigma^I)}$$

plug in $\sigma^I = \delta^I_i(x_i - \phi_i)$ & keep lowest order in ϕ

equal

$$-\left(\delta^I_j - \cancel{\delta^I_k \partial_j \phi_k}\right)\left[-\frac{1}{4}\left(\lambda^{IJKL} + \lambda^{JIKL}\right)\underbrace{\partial_i \sigma^J}_{\simeq \delta^J_i}\underbrace{\left(\partial_\ell \sigma^K \partial_\ell \sigma^L - \delta^{KL}\right)}_{\phi + \cancel{\phi^2}}\right]$$

$$T_{ij} = \frac{1}{2}\lambda_{ij}{}^{KL}\left(-\delta^K_k \delta^L_\ell\right)\left(\partial_k \phi_\ell + \partial_\ell \phi_k\right) = -\lambda_{ijk\ell} u_{k\ell}$$

↳ stress tensor linearly proportional to strain: elastic solid

Stability of equilibrium? $\lambda^{\underline{IJ}\,\underline{KL}} \geq 0$ (positive semidefinite)

analogue of Hooke's Law: $F = -kx$ $(k > 0)$

Isotropic solid: $\lambda_{ijk\ell} = \tilde{K}\delta_{ij}\delta_{k\ell} + \mu\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right)$

Potential energy of deformed solid:

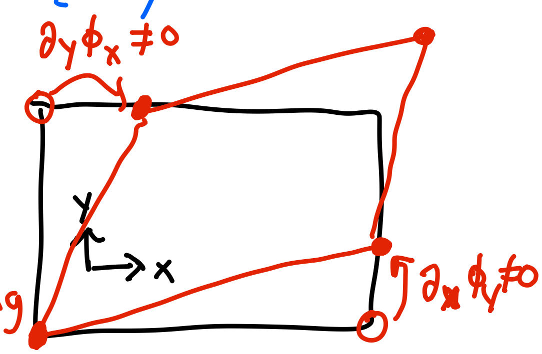
↳ $E_{pot} \sim \frac{1}{2}\lambda_{ijk\ell} u_{ij} u_{k\ell} \geq 0$ $\left(U = \frac{1}{2}kx^2\right)$

Check positivity?

① $u_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

shear

volume-preserving



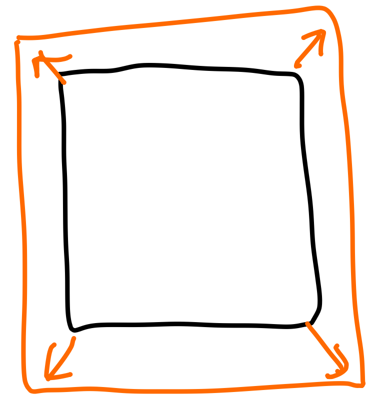
$\lambda_{ijkl} u_{ij} u_{kl} \geq 0 \rightsquigarrow \lambda_{xyxy} + \lambda_{xyyx} + \lambda_{yxxy} + \lambda_{yxyx} \geq 0$

$= 4\mu$. So $\underline{\mu \geq 0}$

(2) Volume-changing:

$u_{ij} = \delta_{ij}$

expansion / compression



$\lambda_{ijkl} \delta_{ij} \delta_{kl} \geq 0$

$\hookrightarrow \tilde{K} \underbrace{\delta_{ij}\delta_{ij}}_{= tr(\mathbb{1}) = 3} \delta_{kl}\delta_{kl} + \mu \delta_{ij}\delta_{kl}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

$\hookrightarrow = 9\tilde{K} + 6\mu \geq 0.$

Write $K = \tilde{K} + \frac{2}{3}\mu$ so that $K \geq 0$.

Now: $T_{ij} = -\lambda_{ijkl} u_{kl}$ where

$\lambda_{ijkl} = K \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$

bulk modulus (compression)

$\hookrightarrow$ shear modulus

projects onto traceless symmetric matrices

With rotation invariance: $u_{ij} \to$ 3x3 matrix (symmetric)

$u_{ij} = u_{ji}$

3x3 matrix (rot. inv.):

$\underset{i}{(\text{spin-}1)} \otimes \underset{j}{(\text{spin-}1)} \to \underbrace{\text{spin-}0}_{\substack{\delta_{ij} \\ \text{(trace)}}} \oplus \underbrace{\text{spin-}1}_{\substack{\text{antisym. matrix} \\ \text{(not allowed)}}} \oplus \underbrace{\text{spin-}2}_{\text{traceless sym.}}$