PHYS 5210
Graduate Classical Mechanics
Fall 2024

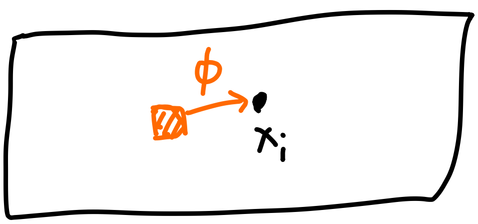
Lecture 19
Sound waves in solids

October 9

"series expansion near eq"

EFT for a solid: $\sigma^I(x_i, t) = \delta^I_i(x_i - \phi_i)$

"chunk" — spatial pos. — displacement field



Last time:

$$\mathcal{L} = \frac{\rho}{2}\partial_t\phi_i\partial_t\phi_i - \frac{1}{2}\lambda_{ijkl}\partial_i\phi_j\partial_k\phi_l$$

where $\lambda_{ijkl} = \lambda_{klij} = \lambda_{jikl}$

For isotropic solid:

$$\lambda_{ijkl} = \underbrace{K\,\delta_{ij}\delta_{kl}}_{\substack{\text{(bulk)}\\ \text{(vol. - changing)}}} + \underbrace{\mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)}_{\substack{\text{(shear)}\\ \text{(vol. - preserving)}}}$$

Stability of solid requires: $K,\ \mu \geq 0$.

Today: normal modes (sound waves) of a solid:

EOMs: (Euler-Lagrange):

$$\frac{\delta S}{\delta \phi_i} = 0 = \cancel{\frac{\partial \mathcal{L}}{\partial \phi_i}}^{\,0} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \phi_i)}$$

$$- \partial_t (\rho \partial_t \phi_i) - \partial_j \left(- \lambda_{jik\ell} \partial_k \phi_\ell \right)$$

Plug in plane wave ansatz: $\phi_i = a_i e^{i[\vec{k}\cdot\vec{x} - \omega t]}$ (a_i const.)

Using rotational invariance: $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$ (choosing coord axes aligned w/ $\vec{k}$)

$\partial_t \phi_i \to -i\omega \phi_i$ $\quad (\partial_x, \partial_y)\phi_i \to 0$ $\quad \partial_z \phi_i \to ik\phi_i$

Plug in to EOM:

$$0 = \rho\omega^2 \phi_i - k_j \lambda_{jik\ell} k_k \phi_\ell = \rho\omega^2\phi_i - k_i K k_\ell \phi_\ell - \mu\left[k^2\phi_i + k_i k_j \phi_j - \tfrac{2}{3} k_i k_j \phi_j\right]$$

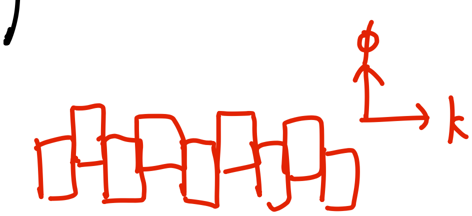
↳ write out as a matrix:

$$\rho\omega^2 \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \underline{\mu k^2} & 0 & 0 \\ 0 & \underline{\mu k^2} & 0 \\ 0 & 0 & \underline{(K + \frac{4}{3}\mu)k^2} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

transverse waves: $\vec{\phi} \perp \vec{k}$ (S-waves)

$\omega = \pm v_s k$

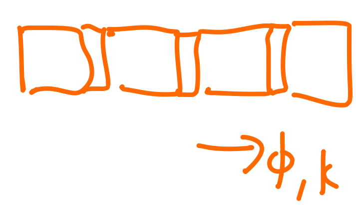
where $v_s = \sqrt{\frac{\mu}{\rho}}$



longitudinal waves: $\vec{\phi} \parallel \vec{k}$ (P-waves)

$\omega = \pm v_p k$

where $v_p = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$



Bulk solids (isotropic) have $\underline{2}$ sound modes (S & P)

→ in liquid, only have a single sound wave (P)

Anisotropic solids: λ_{ijkl} is invariant under point group of crystal

↳ mix up sound modes (esp. S waves)

Back to isotropic solid:

$$\sqrt{\frac{K}{\mu}+\frac{4}{3}} = \frac{v_P}{v_S} \geq \sqrt{\frac{4}{3}}$$

For typical crystalline solid: $\sim 50 m_p$

$$\rho \sim \frac{m_{atom}}{d_{atom}^3} \sim \frac{10^{-25}\ \text{kg}}{(3\times 10^{-10}\ \text{m})^3} \sim 3\times 10^3 \frac{\text{kg}}{\text{m}^3} = 3\rho_{H_2O}$$

(binding E of hydrogen)

$$K, \mu \sim \frac{E_{bond}}{d_{atom}^3} \sim \frac{10^{-18}\ \text{J}}{(3\times 10^{-10}\ \text{m})^3} \sim 3\times 10^{10}\ \text{J}/\text{m}^3$$

$\sim 3 \times 10^5$ atm (of pressure)

$v_S^2 \rho = \mu$

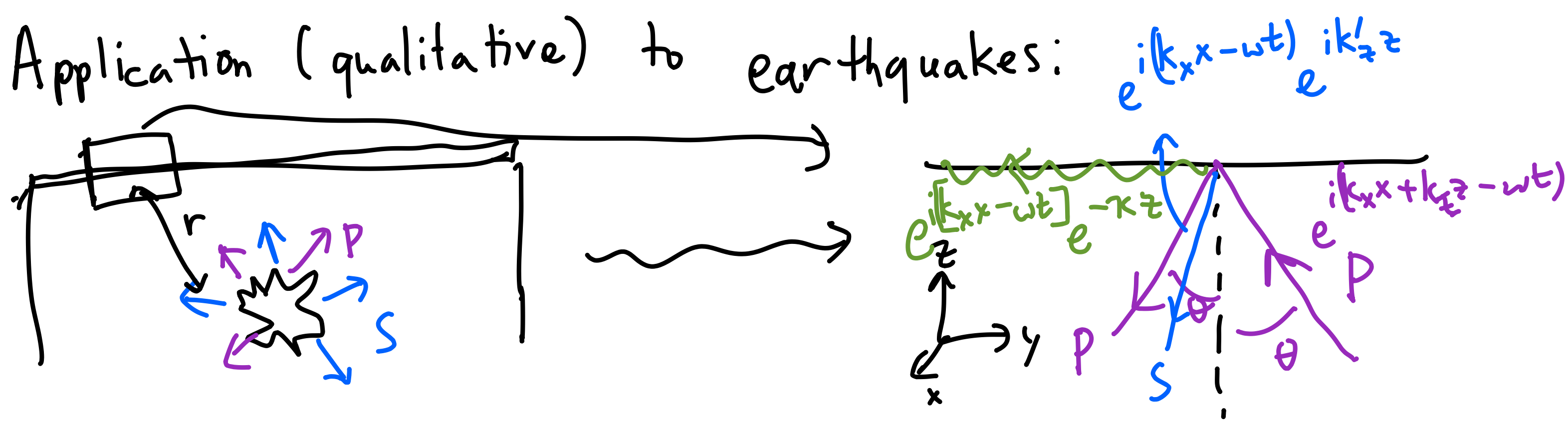
$$[\text{speed}]^2 \frac{[m]}{[L]^3} = \frac{[\text{energy}]}{[L]^3}$$

Combine: $v_S, v_P \sim \sqrt{\frac{3\times 10^{10}}{3\times 10^3}}\ \frac{\text{m}}{\text{s}} \sim 3\ \frac{\text{km}}{\text{s}}$

$\sim 10\times >$ speed of sound in air

$\sim 2-3\times >$ ... water

Application (qualitative) to earthquakes:



At surface of solid: Boundary conditions on
ϕ_x, ϕ_y, ϕ_z [usually: "no-stress": $T_{zj} = 0$]

$$v_s \sqrt{k_x^2 + k_z'^2} = v_p \sqrt{k_x^2 + k_z^2}.$$

Surface waves also exist! Most dangerous

→ bulk S & P sound modes:
amplitude $\sim 1/r$: Power $\sim$ amplitude^2
Power × Area $\sim$ const.

→ surface: Power × $(2\pi r) \sim$ const.

↳ amplitude $\sim 1/\sqrt{r}$.