PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 20
Hamiltonian mechanics

October 11

Why Lagrangian mechanics? $\rightsquigarrow$ Hamiltonian mechanics.

1. deal w/ new configuration spaces (rigid body dynamics)
2. easy to incorporate symmetry
3. Noether's Thm

1. phase space (bit more abstract)
2. also easy here. AND manifest symmetry structure (lec 21–22)
3. symmetry $\Longleftrightarrow$ cons. law (if and only if)

Difference? explicitly first order equations...

coordinates ξ^α on phase space, and

$$\dot{\xi}^\alpha = f^\alpha(\xi)$$

but better!

$$\left[m\ddot{x} = -\frac{\partial U}{\partial x} \right.$$

$$\left. \begin{aligned} v &= \dot{x} \\ m\dot{v} &= -\frac{\partial U}{\partial x} \end{aligned} \right]$$

Today: deducing something Hamiltonian from Lagrangian mechanics

$$S[x_i] = \int dt\, L(x_i, \dot{x}_i) \qquad \text{(first derivatives only!)}$$

$$\frac{\delta S}{\delta x_i} = 0 = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\boxed{\frac{\partial L}{\partial \dot{x}_i}}$$

Define as p_i

$$0 = \frac{\partial L(x,p)?}{\partial x_i} - \dot{p}_i$$

first order?

Now: $\dot{x}_i = f(x,p)$? w/ phase space (x_i, p_i)...

Strategy to change vars from $(\dot{x}_i, x_i)$ to (p_i, x_i): Legendre transformation.

$L(x_i, \dot{x}_i)$

$$dL = \frac{\partial L}{\partial x_i} dx_i + \frac{\partial L}{\partial \dot{x}_i} d\dot{x}_i = \frac{\partial L}{\partial x_i} dx_i + \underbrace{p_i\, d\dot{x}_i}_{= d(p_i \dot{x}_i) - \dot{x}_i dp_i}$$

$L(x_i, \dot{x}(x,p))$

$$d(\underbrace{L - p_i \dot{x}_i}_{-H(x_i,p_i)}) = \frac{\partial L}{\partial x_i} dx_i - \dot{x}_i\, dp_i = -\frac{\partial H}{\partial x_i} dx_i - \frac{\partial H}{\partial p_i} dp_i$$

the Hamiltonian

$$\frac{\partial L}{\partial x_i} = \left[\dot{p}_i = -\frac{\partial H}{\partial x_i}\right] \qquad \dot{x}_i = \frac{\partial H}{\partial p_i}$$

So, if we "invert" $\dot{x}$ (express $\dot{x}(x,p)$ instead of $p(x,\dot{x})$), then by defining $H = p_i \dot{x}_i - L$ ... first-order equations!

Recap: start w/ Lagrangian $L(x_i, \dot{x}_i)$:

① Define $p_i = \dfrac{\partial L}{\partial \dot{x}_i}$ (no Noether)

"canonical momentum"

② Hamiltonian ... obtain by invert $p_i = \frac{\partial L}{\partial \dot{x}_i}(x, \dot{x}) \to \dot{x}(x,p)$

$$H = p_i \dot{x}_i - L$$

$$H = p_i \, \dot{x}_i(x,p) - L(x, \dot{x}(x,p))$$

③ Euler-Lagrange $\longrightarrow$ Hamilton's equations: <u>first-order</u>

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

Define: <u>phase space</u>: coordinates $(x_i, p_i) \in \mathbb{R}^{2n}$ (phase space)
if $x_i \in \mathbb{R}^n$ (config. sp.)

Example: harmonic oscillator:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

① $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

② $\dot{x} = p/m$ and $H = p\frac{p}{m} - \frac{m}{2}\left(\frac{p}{m}\right)^2 + \frac{k}{2}x^2 = \frac{p^2}{2m} + \frac{kx^2}{2}$

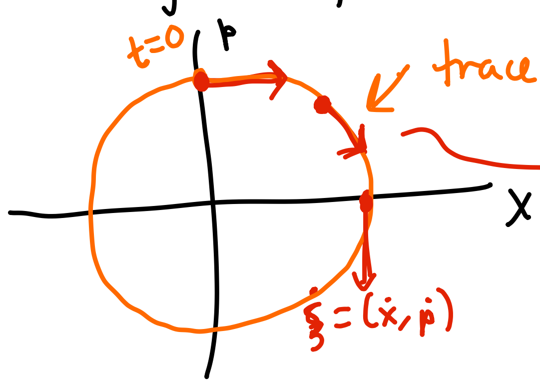
③ $\dot{x} = \frac{p}{m} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial x} = -kx$

Combine to reproduce E-L equations:

$$\dot{p} = \frac{d}{dt}(m\dot{x}) = m\ddot{x} = -kx \longrightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

From this perspective, nothing new?

Advantage: dynamics as flow on phase space!



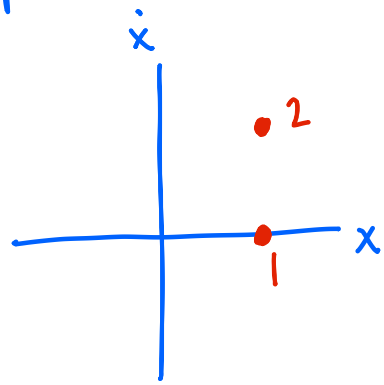
trace out trajectory thru phase space.

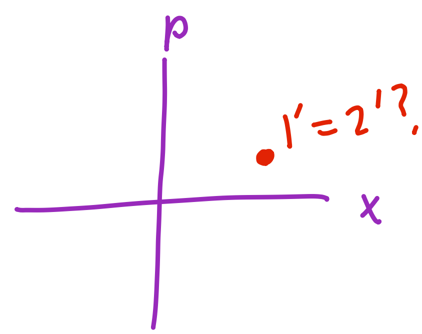
drawn "Hamiltonian vector field", which "generates time-translation" in phase space.

Given a Lagrangian $L(x_i, \dot{x}_i)$, can I always find H?

Need Legendre transform to exist.

config space $(x_i, \dot{x}_i) \xrightarrow{\; p_i = \frac{\partial L(x,\dot{x})}{\partial \dot{x}_i} \;}$ phase space (x_i, p_i)





Bad?

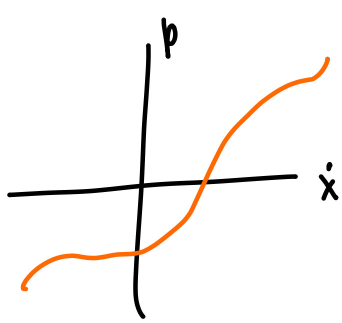
$$L(x,\dot{x}) \to \underbrace{H(x,p) \to L(x,\dot{x})}$$

$$d(\underbrace{H - p_i \dot{x}_i}_{-L}) = \frac{\partial H}{\partial x_i} dx_i - p_i d\dot{x}_i \quad \text{(using Ham's eqs)}$$

So Legendre transform must be invertible:

Need $p_i = \frac{\partial L}{\partial \dot{x}_i}$ to have unique solutions, or

$\frac{\partial L}{\partial \dot{x}_i}(x, \underline{\dot{x}})$ monotonically increasing (or decreasing) w/ $\dot{x}$.



$p = f(\dot{x})$ & $\dot{x} = f^{-1}(p)$ well-defined!

Practical check: $\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} > 0$ (Hessian positive-def.)

Some Lag systems not Ham / some Ham not Lag.