PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 21
Poisson brackets

October 14

Review: "canonical" $\underbrace{\text{phase space}}_{\mathbb{R}^{2n}:\ (x_1,\ldots,x_n,\ p_1,\ldots,p_n)}$

$\underbrace{p_1,\ldots,p_n}$ e.g. $p_i = \frac{\partial L(x,\dot{x})}{\partial \dot{x}_i}$

Legendre transform from Lagrangian $\rightarrow$ $\underbrace{\text{Hamiltonian}}$ mechanics:

$H = p_i \dot{x}_i - L$

EOMs (Hamilton's eqs) $\dot{x}_i = \frac{\partial H}{\partial p_i}$ & $\dot{p}_i = -\frac{\partial H}{\partial x_i}$

Today: define Poisson bracket & symmetry algebras...

Observation:

$$\underbrace{\frac{d}{dt}F(t,x_i,p_i)}_{\text{following } F \text{ along phys. traj.}} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = \frac{\partial F}{\partial t} + \underbrace{\frac{\partial F}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x_i}}$$

"generator of time evolution" $\{F,H\} = \frac{\partial F}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x_i}\frac{\partial F}{\partial p_i}$

"Intrinsically Hamiltonian": $\dot{F} = \{F,H\}\ (+\partial_t F)$

Properties of Poisson brackets:

① Antisymmetry: $\{A,B\} = -\{B,A\}$

② Linearity: $\{\alpha_1 A_1 + \alpha_2 A_2, B\} = \alpha_1\{A_1,B\} + \alpha_2\{A_2,B\}$.

const.

③ Invertibility: for coordinates $\xi^\alpha = \begin{pmatrix}\vec{x}\\ \vec{p}\end{pmatrix}$ on phase space

$\{\xi^\alpha, \xi^\beta\} = V^{\alpha\beta}$ is an invertible matrix

inverses

Define: symplectic form: $\omega_{\alpha\beta}$ such that $\omega_{\alpha\beta}V^{\beta\gamma} = \delta^\gamma_\alpha$.

Example 1: canonical PBs are invertible:

$$\{x_i, p_j\} = \frac{\partial x_i}{\partial x_k}\frac{\partial p_j}{\partial p_k} - \cancel{\frac{\partial x_i}{\partial p_k}}\,\cancel{\frac{\partial p_j}{\partial x_k}} = \delta_{ik}\delta_{jk} = \boxed{\delta_{ij}}$$

$$\{x_i, x_j\} = \{p_i, p_j\} = 0.$$

$$V = \begin{array}{cc} \begin{array}{cc} x & p \end{array} & \\ \left(\begin{array}{c|c} 0 & \delta_{ij} \\ \hline -\delta_{ij} & 0 \end{array}\right) & \begin{array}{c} x \\ p \end{array} \end{array} \qquad \omega = \left(\begin{array}{c|c} 0 & -\delta_{ij} \\ \hline \delta_{ij} & 0 \end{array}\right)$$

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④ Jacobi identity: for any functions F, G, H:

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0.$$

Why: suppose F, G, H are all t-ind. Then

$$\dot{F} = \{F,H\} \qquad \dot{G} = \{G,H\}$$

$$\frac{d}{dt}\{F,G\} = \{\dot{F},G\} + \{F,\dot{G}\} \quad (\text{by } ②)$$

$$\{\{F,G\},H\} = \{\{F,H\},G\} + \{F,\{G,H\}\}$$

$$= -\{\{H,F\},G\} - \{\{G,H\},F\} \qquad \text{by } ①$$

Jacobi allows $\dot{F} = \{F, H\}$ to be consistent!

Properties ①, ②, ④ $\mapsto$ PBs form Lie algebra.

For us: if $F^1, F^2, \ldots$ (F^a) is a basis of functions:

$$\{F^a, F^b\} = \underbrace{f^{abc}}_{\text{structure constants.}} F^c$$

structure constants. (don't depend on x, p)

Closed (sub)algebra: finite list of $F^1, \ldots, F^m$ s.t.

$$\{F^a, F^b\} = f^{abc} F^c \qquad a, b, c = 1, \ldots, m$$

↳ lec 22: symmetries...

Example 2: Consider $F^1 = x$, $F^2 = p$, $F^3 = 1$: (canonical PB)

$$\{F^1, F^2\} = F^3 \qquad \{F^1, F^3\} = \{F^2, F^3\} = 0$$

So $f^{123} = -f^{213} = 1$ $\qquad f^{132} = f^{231} = \cdots = 0.$

F^1, F^2, F^3 form a closed algebra.

Proposition: in a Hamiltonian system $[\dot{F} = \{F, H\}]$, then conserved quantities form closed Lie (symmetry) algebra.

Why? $\dot{F} = 0$ and $\dot{G} = 0$

$$\frac{d}{dt}\{F, G\} = 0 \quad \text{by Jacobi identity.}$$

Contrast w/ Lagrangian mechanics? (coord transform leaving S invariant), algebraic structure less obvious.

Hamiltonian: cons. Q's form algebra (generate group)

↳ symmetry!

Example 3: rotation symmetry

Consider phase space $\mathbb{R}^6 = (\underbrace{x_1, x_2, x_3}, \underbrace{p_1, p_2, p_3})$

$x_i \to Q_{ij} x_j$ $\quad$ $p_i \to Q_{ij} p_j$

(Q orthogonal)

conserved Q: angular momentum $L_i = \varepsilon_{ijk} x_j p_k$

($\vec{L} = \vec{x} \times \vec{p}$)

From Hamiltonian perspective ... (w/o coord transforms?)

↳ If L_x, L_y, L_z ... any other conservation laws?

Need to calculate: $\{L_i, L_j\} = \{L_i, \varepsilon_{jkl} x_k p_l\}$

$$= \varepsilon_{jkl} x_k \{L_i, p_l\} + \varepsilon_{jkl} p_l \{L_i, x_k\}$$

$$\{L_i, x_k\} = \varepsilon_{imn} \{x_m p_n, x_k\} = \varepsilon_{imn} \left[\{x_m, x_k\} p_n + x_m \{p_n, x_k\} \right]$$
$$= \varepsilon_{imn} \left[0 - x_m \delta_{nk} \right]$$
$$= -\varepsilon_{imk} x_m = \varepsilon_{ikm} x_m$$

(Note: looks like rotation around i-axis)

Similarly: $\{L_i, p_k\} = \varepsilon_{ikm} p_m$

$\varepsilon_{ijk} \varepsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}$

Now: $\{L_i, L_j\} = \varepsilon_{jkl} \left[x_k \varepsilon_{iln} p_n + p_l \varepsilon_{ikn} x_n \right]$

$$= -(\delta_{ij} \delta_{kn} - \delta_{ik} \delta_{jn}) x_k p_n + (\delta_{ij} \delta_{ln} - \delta_{il} \delta_{jn}) p_l x_n$$
$$= -\delta_{ij} x_k p_k + x_i p_j + \delta_{ij} x_l p_l - p_i x_j$$
$$= x_i p_j - p_i x_j = \varepsilon_{ijk} L_k$$

Therefore: angular momenta form closed algebra.
OK to have just rotational symmetry.
form of PBs matches SO(3) ...