PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 22
Canonical transformations

October 16

What is a $\underbrace{\text{symmetry}}$ in Hamiltonian mechanics?

Lec 23: is a canonical transformation (CT) that leaves H invariant.

?? Goal for today.

CT = a coord transform that leaves $\underbrace{\text{Poisson brackets}}$ invariant

in Ham mechanics, PBs conceptually independent of H. PBs are more important.

index: 1, …, 2n

Start w/ coords $\begin{pmatrix} x_i \\ p_i \end{pmatrix} \to \xi^\alpha$. CT is $\xi^\alpha \to \eta^\alpha(\xi)$ s.t.

$$\{\xi^\alpha, \xi^\beta\} = \{\eta^\alpha(\xi), \eta^\beta(\xi)\}$$

For now: "canonical" coordinates: $\{x_i, x_j\} = \{p_i, p_j\} = 0$

$$\boxed{\{x_i, p_j\} = \delta_{ij}.}$$

canonical conjugates.

So CT: $\eta^\alpha = \begin{pmatrix} X_i \\ P_i \end{pmatrix}$: $\{X_i, P_j\} = \delta_{ij}$
$\{X_i, X_j\} = \{P_i, P_j\} = 0.$

Note: in new coordinates, Ham's eq's "look same":

$\dot{F} = \{F, H\}$ (if F t-ind.)

$\hookrightarrow = \frac{\partial F}{\partial x_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x_i}$ or $= \frac{\partial F}{\partial X_i}\frac{\partial H}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial H}{\partial X_i}$

$F = X_i$: $\dot{X}_i = \frac{\partial H}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial H}{\partial X_i}$ ← write $H(x(X,P), p(X,P))$

Example 1: check canonicity?

A) $X = x+p$, $P = p$: $\{X, P\} = \{x+p, p\} = \{x,p\} + \underbrace{\cancel{\{p,p\}}}_{0} = 1$ ✓

B) $X = x^2$, $P = p$: $\{X, P\} = \frac{\partial(x^2)}{\partial x}\frac{\partial p}{\partial p} - \cancel{\frac{\partial(x^2)}{\partial p}}\cancel{\frac{\partial p}{\partial x}} = 2x$ ✗

C) $X = 3x$, $P = p$: $\{X, P\} = 3 \neq 1$

Can we classify all of the $\wedge$ CTs:

infinitesimal: $\xi^\alpha \to \xi^\alpha + \varepsilon\,\theta^\alpha + \dots$

Recall some definitions:

$$\{\xi^\alpha, \xi^\beta\} = V^{\alpha\beta}$$

$$\begin{array}{cc} \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right) & \\ x \quad\; p & \end{array} \leftarrow \{x_i, p_j\} = \delta_{ij} \dots$$

$$V^{-1} = \omega = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right) \quad (\omega_{\alpha\beta})$$

(symplectic form)

check CT?

$$\{\xi^\alpha + \varepsilon\theta^\alpha, \xi^\beta + \varepsilon\theta^\beta\} = \{\xi^\alpha, \xi^\beta\} + \varepsilon\Big[\overbrace{\{\theta^\alpha, \xi^\beta\} + \{\xi^\alpha, \theta^\beta\}}^{=0\,?}\Big] + \dots$$

Now: $\{\theta^\alpha, \xi^\beta\} = \frac{\partial \theta^\alpha}{\partial \xi^\gamma} V^{\gamma\delta} \frac{\partial \xi^\beta}{\partial \xi^\delta} = \frac{\partial \theta^\alpha}{\partial x_i}\frac{\partial \xi^\beta}{\partial p_i} - \dots$

$$= \partial_\gamma \theta^\alpha V^{\gamma\delta} \delta^\beta_\delta = V^{\gamma\beta} \partial_\gamma \theta^\alpha$$

So: $0 = V^{\gamma\beta}\partial_\gamma \theta^\alpha - V^{\gamma\alpha}\partial_\gamma \theta^\beta$

↳ multiply by $\omega_{\beta\beta'}\omega_{\alpha\alpha'}$ (lower indices using ω):

$$0 = \delta^\gamma_{\beta'}\partial_\gamma(\theta^\alpha \omega_{\alpha\alpha'}) - \delta^\gamma_{\alpha'}\partial_\gamma(\underbrace{\theta^\beta \omega_{\beta\beta'}}_{=\theta_{\beta'}}) \to \underbrace{0 = \partial_{\beta'}\theta_{\alpha'} - \partial_{\alpha'}\theta_{\beta'}}$$

Math fact: ("Helmholtz decomposition"/ dR cohomology): on phase space $\mathbb{R}^{2n}$, all sol'ns are $\theta_\alpha = \partial_\alpha F$ ← function

So: all infinitesimal CTs: $\xi^\alpha \to \xi^\alpha + \varepsilon V^{\alpha\beta}\partial_\beta F$

$$= \xi^\alpha + \varepsilon\{\xi^\alpha, F\}.$$

Every function generates infinitesimal CT F. (And there aren't any others)

Example 2: Take phase space $\mathbb{R}^{2n}$, consider CT generated by (one of) momentum p_1:

$$\xi^\alpha \to \xi^\alpha + \varepsilon\{\xi^\alpha, p_1\}$$

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \to \begin{pmatrix} x_i \\ p_i \end{pmatrix} + \varepsilon \begin{pmatrix} \{x_i, p_1\} \\ \{p_i, p_1\} \end{pmatrix} = \begin{pmatrix} x_i + \varepsilon \delta_{i1} \\ p_i \end{pmatrix}$$

Translation is generated by momentum.

Example 3: rotation on phase space $\mathbb{R}^6$ (3 spatial dim).

Last time: $\{x_i, L_j\} = -\varepsilon_{ijk}x_k$ and $\{p_i, L_j\} = -\varepsilon_{ijk}p_k$.

Claim: L_j generates rotation around j-axis.

Example: $j=z$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x+\varepsilon y \\ y-\varepsilon x \\ z \end{pmatrix} \qquad \{x, L_z\} = -\varepsilon_{xzy} x_y = -(-1)\, y$$

momentum rotates w/ same orientation!

Example 4: time-translation.

Recall: $\dot{F} = \{F, H\}$ (assuming $F(\xi)$)

↳ or: $F(t=\varepsilon) = F(0) + \varepsilon \dot{F}(0) + \dots = F(0) + \varepsilon \{F, H\}$

So H generates CTs $\sim \begin{pmatrix} x \\ p \end{pmatrix} \to \begin{pmatrix} x(\varepsilon) \\ p(\varepsilon) \end{pmatrix}$

Keep generating CTs:

$$\xi^\alpha \xrightarrow{CT} \xi^\alpha(t=\varepsilon) \xrightarrow{CT} \xi^\alpha(t=2\varepsilon) \xrightarrow{CT} \cdots \xrightarrow{CT} \xi^\alpha\left(\frac{t}{\varepsilon}\cdot\varepsilon\right) = \xi^\alpha(t)$$

Define "adjoint action": $\mathrm{ad}_F g = \{F, g\} = -\{g, F\}$.

Formally solve Ham's eqs: $\xi^\alpha(t) = e^{-t\cdot \mathrm{ad}_H} \xi^\alpha$ ← operator.

↳ $$\frac{d}{dt}\xi^\alpha(t) = -\mathrm{ad}_H\left(e^{-t\cdot \mathrm{ad}_H}\xi\right) = \{\xi^\alpha(t), H\}.$$

Analogy to QM: Heisenberg picture: for any operator A,

$$A(t) = e^{iHt/\hbar} A e^{-iHt/\hbar}$$

If we defined $\mathrm{ad}_H A = [H, A]$, (matrix commutator)

then $$A(t) = e^{\frac{it}{\hbar}\cdot \mathrm{ad}_H} A$$