PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 23
Noether's Theorem in Hamiltonian mechanics

October 18

New perspective from principle of least action:

$$S = \int dt\, L(x_i, \dot{x}_i) = \int dt \left[p_i \dot{x}_i - H(x,p)\right]$$

$$\hookrightarrow \frac{\delta S}{\delta x_i} = 0 = -\frac{\partial H}{\partial x_i} - \frac{d}{dt} p_i \quad \& \quad \frac{\delta S}{\delta p_i} = 0 = \dot{x}_i - \frac{\partial H}{\partial p_i}$$

Reproduces Hamilton's equations!

Canonical transformation (infinitesimal) = symmetry of S w/ $H=0$.

Check: $\quad p_i \to p_i + \varepsilon P_i \qquad x_i \to x_i + \varepsilon X_i$

$$S \to S + \varepsilon \int dt \left(P_i \dot{x}_i + p_i \dot{X}_i\right) = S + \varepsilon \int dt \underbrace{\left(P_i \dot{x}_i - X_i \dot{p}_i\right)}_{\text{need to be } \dot{\Phi}}$$

compare?

$$\frac{d}{dt}\Phi(x,p) = \dot{x}_i \frac{\partial \Phi}{\partial x_i} + \dot{p}_i \frac{\partial \Phi}{\partial p_i}$$

$$\frac{\partial \Phi}{\partial x_i} = P_i = -\{p_i, \Phi\} \qquad -\frac{\partial \Phi}{\partial p_i} = X_i = -\{x_i, \Phi\}$$

(X,P) infinitesimal CT!

Is this CT a symmetry when $H \neq 0$?

Since $H(x,p)$ doesn't depend on $\dot{x}$ & $\dot{p}$, so if $L \to L + \epsilon \dot{\Phi}$, then H must be invariant: $H(x+\epsilon X, p+\epsilon P) = H(x,p) + O(\epsilon^2)$

$$H = H + \epsilon \left[X_i \frac{\partial H}{\partial x_i} + P \frac{\partial H}{\partial p_i} \right] \quad \text{or} \quad 0 = -\frac{\partial \Phi}{\partial p_i}\frac{\partial H}{\partial x_i} + \frac{\partial \Phi}{\partial x_i}\frac{\partial H}{\partial p_i}$$

$$0 = \{\Phi, H\}$$

Reminder: Φ is conserved if $\dot{\Phi} = 0 = \{\Phi, H\}$ (if Φ t-ind.)

Noether's Thm in Hamiltonian mechanics:

1. If CT generated by Φ is symmetry, $\{\Phi, H\} = 0 \Rightarrow \Phi$ conserv.
2. $\underbrace{\text{If } \Phi \text{ conserved,}}_{\text{new!}}$ $\{\Phi, H\} = 0$, then S "invariant" under CT generated by Φ.

Now: symmetry <u>if and only if</u> conserved quantity.

What's Noether charge (from lec 3)? $Q = \pm \Phi$

Analogy to QM:

continuous symmetry?

$U^\dagger H U = H$ $\longrightarrow$ $[F, H] = 0$.

unitary

$U = e^{-i\epsilon F}$

infinitesimal

Hermitian (observable)

$\hookrightarrow$ F is conserved!

$\{\Phi, H\} = 0$

Φ generated symmetry

Example 1: collection of symmetries...

lec 22: • translation sym. $x \to x + \epsilon$, generated by CT using momentum p

Therefore if $H(x) = H(x+\varepsilon)$, $\left[\{p, H\} = 0 \right.$, and p is conserved.
then infinitesimal CT generated by p leaves H inv.... $\left. H(x,p) = H(x+\varepsilon, p) \right]$

- rotation symmetries generated by angular momentum L_i.

Symplectic Reduction: given some Hamiltonian system (phase space & PBs): then find lower-dimensional phase space after "mod out" by a continuous sym.

Pick function F, suppose $F=0$ is a "smooth" $(2n-1)$-dim. subspace of $\mathbb{R}^{2n}$.

Take $\{F=0\} \subseteq \mathbb{R}^{2n}$, identify point ξ^α w/ $e^{s \cdot \mathrm{ad}_F} \xi^\alpha$

↳ $(2n-2)$-dim space is "Hamiltonian" (symplectic manifold) ← Lec 24

Symmetry let's you lower dimension of phase space by 2.

Example 2: translation symmetry.

Take phase space $\mathbb{R}^4$, canonical coords (x_1, x_2, p_1, p_2)

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1)$$

Symmetry: $H(x_2+\varepsilon) = H(x_2)$ [x_2-trans sym]

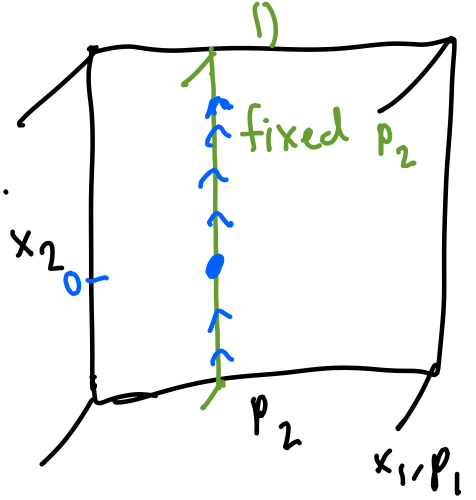
↓

CT generated by p_2.

Symplectic reduction: phase space (x_1, p_1).

1) Fix $p_2 = 0$ (or any other const.) [OK, p_2 conserved]

2) CT generated by p_2: $\begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2+\varepsilon \\ p_1 \\ p_2 \end{pmatrix}$ } identify these pts!



symplectic reduction to $\mathbb{R}^2$. $\rightarrow$

Math: $\mathbb{R}^2$ also has natural PB "inherited from $\mathbb{R}^4$'s"

$\rightarrow$ guess: $\{x_1, p_1\} = 1$

Physics: 1 symmetry/ Cons. law p_2 $\longrightarrow$ "remove" both p_2 & x_2.

canonical conj.

$$H = \left[\frac{p_1^2}{2m} + V(x_1)\right] + \frac{p_2^2}{2m}$$

"reduced H"

const. offset

"forgotten"?

$\dot{p}_2 = 0$

$\dot{x}_1 = \frac{\partial H}{\partial p_1}$ only has x_1, p_1 dynamical

$\dot{p}_1 = -\frac{\partial H}{\partial x_1}$

$\dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m}$

$x_2(t) = x_2(0) + \frac{p_2 t}{m}$

In generality:

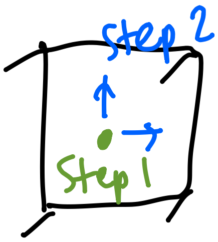
$$\dot{x}_2 = \frac{\partial H(x_1, p_1, p_2)}{\partial p_2}$$

Noether: "integration const."

If solved for x_1, p_1, x_2 & p_2 "completely determined"

Example 3: non-commuting symmetries?

Suppose $H(x_1, p_1)$ only, so x_2 & p_2 generate symmetries.



$\rightsquigarrow$ same reduced phase space $\mathbb{R}^2$.

Example 4: rotation sym / central force problem:

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$\downarrow$$

$$H = \frac{p_r^2}{2m} + \left[\frac{p_\theta^2}{2mr^2} + V(r)\right]$$

$V_{eff}(r)$: "central force reduction"

Symplectic reduction by $\theta \to \theta + \varepsilon$ trans. symmetry

aka symmetry generated by p_θ.

Study Ham mech on phase space (r, p_r).