PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 24
Symplectic geometry

October 21

Hamiltonian mechanics so far:

1. start w/ phase space $\mathbb{R}^{2n}$: $(x_1, \dots, x_n, p_1, \dots, p_n)$
2. Define Poisson Bracket (symplectic form): $\{x_i, p_j\} = \delta_{ij}$
3. Define time evolution as CT generated by Hamiltonian H:

$$S = \int dt \left[p_i \dot{x}_i - H\right]$$

In Lagrangian mechanics, could also study config spaces beyond $\mathbb{R}^n$. What about phase space beyond $\mathbb{R}^{2n}$?

Hamiltonian mechanics defineable on symplectic manifold (M, ω):

→ M is $2n$-dim manifold (smooth space, "calculus OK")

→ ω is the symplectic form; ω is closed:

$\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ is invertible $\quad \partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta} = 0$

Both M & ω necessary.

Why ω closed/invertible?

↳ intuitive: Poisson bracket exists: $\{\xi^\alpha, \xi^\beta\} = V^{\alpha\beta}$.

Proposition: if ω is closed and $\omega^{-1} = V$, then V obeys Jacobi identity.

Example 1:

Lagrangian system w/ S^1 (circle) configuration space: $L = \frac{1}{2}\dot{\theta}^2$

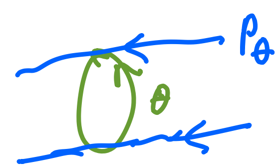
Legendre transform?

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \dot{\theta} \qquad \text{and} \qquad H = p_\theta \dot{\theta} - L = \frac{1}{2} p_\theta^2$$

$\dot{\theta}$ is unbounded

Think about coords: $\theta \sim \theta + 2\pi$ (angular variable) $\qquad$ p_θ unconstrained

Resulting phase space = cylinder $(S^1_\theta \times \mathbb{R}_{p_\theta})$



Formally: $\{\theta, p_\theta\} = 1$?

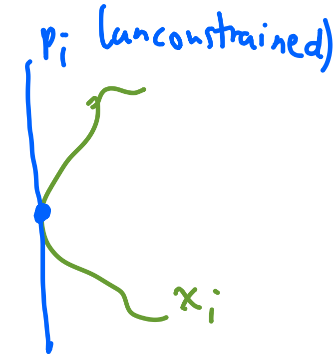
but θ doesn't exist... $\quad \{e^{i\theta}, p_\theta\} = i e^{i\theta}$ e.g.

More general: Lagrangian system has configuration space X.

$\hookrightarrow L(x_i, \dot{x}_i)$ where $x_i \in X$

Phase space M: T^*X = cotangent bundle

$(x_i \in X,\ p_i \in \mathbb{R}^n) \in T^*X$



Like cylinder: finding nice global coords can be hard.

(PBs $\{x_i, p_j\} = \delta_{ij}$ may not make sense in one coord system)

Locally: nice canonical coords always exist. (Darboux's Thm)

Example 2: T^*S^n from symplectic reduction

n-dim sphere

We can embed S^n in $\mathbb{R}^{n+1}$:

$$1 = x_0^2 + x_1^2 + \dots + x_n^2$$

$$S = \int dt \left[L(x_i, \dot{x}_j) + \lambda \underbrace{(x_0^2 + x_1^2 + \dots + x_n^2 - 1)}_{F} \right]$$

Legendre transform?

yes! Dirac bracket prescriptions (not covered here...)

We can guess the output via symplectic reduction.
Start w/ $(x_i, p_i) \in \mathbb{R}^{2n+2}$; reduce by F.

1. Restrict to points where $F = 0$.
2. Identify all pts $\xi^\alpha(s) = e^{s \cdot ad_F}\xi^\alpha$ as a single point.

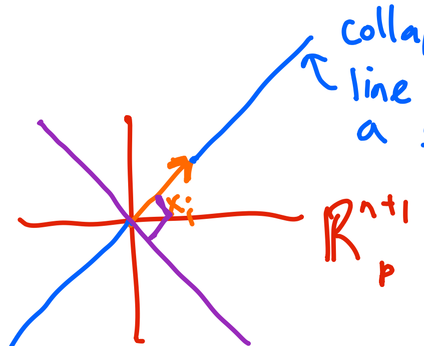
$$\frac{d\xi^\alpha}{ds} = \{F, \xi^\alpha\}:$$

$$\frac{dx_i}{ds} = 0$$

$$\frac{dp_i}{ds} = \{x_i^2, p_i\} = 2x_i$$

$$\underbrace{p_i(s) = p_i + 2s\,x_i}$$

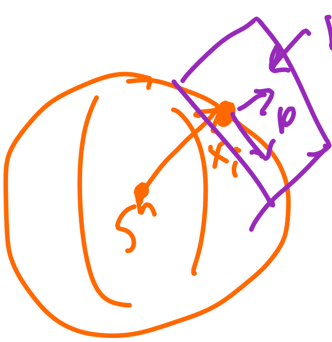
collapse line to a single point



n-perpendicular directions survive: only motion along S^n allowed!

Now our phase space is:

$\mathbb{R}^n$ for momenta



Crudely: near $\bar{x}_i$,

canonical positions:

$$x_i - (\bar{x} \cdot x)\bar{x}_i$$

canonical momenta:

$$p_i - (\bar{x} \cdot p)\bar{x}_i \qquad [p_i - x_j p_j x_i]$$

HW8: you can also find compact symplectic manifolds (S^2)
↳ beyond (usual) Lagrangian mechanics.

We can also build non-trivial symplectic forms on simple phase spaces...

Recall: $S = \int dt\, [\underbrace{p_i \dot{x}_i}_{} - H]$

re-write: $\lambda_\alpha \dot{\xi}^\alpha$

symplectic potential

here: $\lambda_\alpha = \begin{pmatrix} p_i \\ \hline 0 \end{pmatrix}\begin{matrix} x \\ p \end{matrix}$

$(\lambda_{x_i} = p_i$
$\lambda_{p_i} = 0)$.

Generalization? $S = \int dt\, [\lambda_\alpha \dot{\xi}^\alpha - H]$.

(Assume $\partial_t \lambda_\alpha = 0$)

Euler-Lagrange equations:

$$0 = \frac{\delta S}{\delta \xi^\alpha} = \left((\partial_\alpha \lambda_\beta)\dot{\xi}^\beta - \frac{\partial H}{\partial \xi^\alpha}\right) - \frac{d}{dt}\lambda_\alpha$$
$$= \partial_\alpha \lambda_\beta \dot{\xi}^\beta - \partial_\alpha H - \partial_\beta \lambda_\alpha \dot{\xi}^\beta$$

Define $\omega_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha$ as <u>symplectic form</u>:

$$\underbrace{\partial_\alpha H = \omega_{\alpha\beta}\dot{\xi}^\beta}_{\text{general Hamilton's eq.}} \qquad [\, dH = \iota_X \omega \,]$$

If we can invert ω: $\omega^{-1} \to V^{\alpha\beta} = \{\xi^\alpha, \xi^\beta\}$ (locally)

$$\dot{\xi}^\alpha = V^{\alpha\beta}\partial_\beta H = \{\xi^\alpha, H\}$$

If we can't invert ω: some ξ^α are Lagrange multipliers...

Check: all ω derived in this way are closed:

$$\partial_\gamma \omega_{\alpha\beta} + \partial_\alpha \omega_{\beta\gamma} + \cdots = \partial_\gamma(\cancel{\partial_\alpha \lambda_\beta} - \partial_\beta \lambda_\alpha) + \partial_\alpha(\partial_\beta \lambda_\gamma - \cancel{\partial_\gamma \lambda_\beta}) + \cdots = 0.$$

Example 3: phase space $\mathbb{R}^2$

$$S = \int dt \left[(p + p^3)\dot{x} - H(x,p) \right]$$

↳ symplectic potential: $\lambda_x = p + p^3 \qquad \lambda_p = 0$

symplectic form: $\omega_{px} = \partial_p \lambda_x - \partial_x \lambda_p = 1 + 3p^2$

Hamilton's equations:

$$\frac{\partial H}{\partial x} = \omega_{xx}\dot{x} + \omega_{xp}\dot{p} = -\omega_{px}\dot{p} = -(1+3p^2)\dot{p}$$

$$\frac{\partial H}{\partial p} = \omega_{px}\dot{x} = (1+3p^2)\dot{x}$$