PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 25
Rigid body rotation in Hamiltonian mechanics

October 23

Lec 9, 10: $\underbrace{\text{configuration space}}$ of rigid body (one point fixed):

$SO(3)$: R_{iI} = 3x3 $\underbrace{\text{orthogonal matrix}}$ (change of basis)

i: space frame — I: body frame

↓ Lagrange multipliers for constraints

Lagrangian mechanics:

$$L = \frac{1}{2} K_{IJ} \dot{R}_{iI} \dot{R}_{iJ} + \Lambda_{IJ}(R_{iI} R_{iJ} - \delta_{IJ}) + \ldots$$

(free rigid body: left-SO(3) invariance: all space frame $ij\ldots$ contracted)

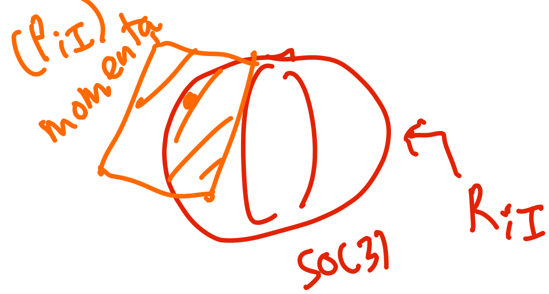
↳ Euler equations: (body frame angular vel. ω_I):

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 \quad \text{etc.}$$

In Hamiltonian mechanics:

① $\underbrace{\text{phase space } M}$ → ② Poisson bracket/ symplectic form ω → ③ Hamiltonian H, generates dynamics.

Lec 24: $T^* SO(3)$ (Cotangent bundle)



Often helpful to write action:

$$S = \int dt \left[\tfrac{1}{2} P_{iI} \dot{R}_{iI} - H + \Lambda(\cdots) \right]$$

↖ useful convention.

But P_{iI} not arbitrary! → only 3 momenta!

Write: $P_{iI} = R_{iJ} P_{JI}$. Claim: $P_{JI} = -P_{IJ}$

lec 10: orthogonality on R gave $R_{iJ}\dot{R}_{iI} = \Omega_{JI} = -\Omega_{IJ}$

$$P_{iI}\dot{R}_{iI} = P_{JI} R_{iJ} \dot{R}_{iI} = P_{JI} \underbrace{\Omega_{JI}}_{\text{antisym}}$$

only antisym. part survives

important

$P_{JI} = -\varepsilon_{JIK} L_K$ → L_K are angular momenta in body frame we will see that

↳ unconstrained.

constrained

Summarize: $M = T^*SO(3)$. coords (R_{iI}, L_I)

What's symplectic form ω? or Poisson brackets ← easier

Strategy: calculate general EoMs... $\dot{L}_I = \{L_I, H\} \ldots$

$$= \{L_I, L_J\} \frac{\partial H}{\partial L_J} + \cdots$$

$$S = \int dt \left[-\tfrac{1}{2} \varepsilon_{JIK} L_K R_{iJ} \dot{R}_{iI} - H(R, L) + \Lambda_{IJ}(R_{iI}R_{iJ} - \delta_{IJ}) \right]$$

$$\frac{\delta S}{\delta L_I} = 0 = -\tfrac{1}{2} \varepsilon_{JKI} R_{iJ} \dot{R}_{iK} - \frac{\partial H}{\partial L_I}$$

$$\frac{\delta S}{\delta R_{iI}} = 0 = -\tfrac{1}{2} \varepsilon_{IJK} \dot{R}_{iJ} L_K - \frac{\partial H}{\partial R_{iI}} + 2\Lambda_{IJ}R_{iJ} + \frac{d}{dt}\left(\tfrac{1}{2} \varepsilon_{JIK} R_{iJ} L_K \right)$$

Remove $\Lambda_{IJ} = \Lambda_{JI}$. Multiply above by R_{iL} → $2\Lambda_{IL}$

since $R_{iJ}R_{iL} = \delta_{JL}$

Take antisymmetric part: $\varepsilon_{ILM}(\cdots)_{IL}$

$$0 = \varepsilon_{ILM}\left(-\varepsilon_{IJK}\dot{R}_{iJ}L_K R_{iL} - \tfrac{1}{2}\varepsilon_{ILK}\dot{L}_K - \frac{\partial H}{\partial R_{iI}}R_{iL}\right)$$

$$= -(\delta_{LJ}\delta_{MK} - \delta_{LK}\delta_{MJ})\dot{R}_{iJ}R_{iL}L_K - \tfrac{1}{2}(2\delta_{MK})\dot{L}_K - \varepsilon_{ILM}R_{iL}\frac{\partial H}{\partial R_{iI}}$$

$$= R_{iK}L_K\dot{R}_{iM} - \dot{L}_M - \varepsilon_{ILM}R_{iL}\frac{\partial H}{\partial R_{iI}}$$

Using: $\frac{1}{2}\varepsilon_{JKI}R_{iJ}\dot{R}_{iK} = -\frac{\partial H}{\partial L_I} \rightsquigarrow -\varepsilon_{JKI}\frac{\partial H}{\partial L_I} = R_{iJ}\dot{R}_{iK}$

Goal: $\dot{R}_{iM} = -\varepsilon_{MNK}\frac{\partial H}{\partial L_N}R_{iK} \rightsquigarrow \{F, H(R_{iI}, L_J)\} = \{F, R_{iI}\}\frac{\partial H}{\partial R_{iI}}$

also: $\dot{L}_M = -\varepsilon_{MNK}\frac{\partial H}{\partial L_N}L_K - \varepsilon_{MIL}\frac{\partial H}{\partial R_{iI}}R_{iL}$ $\quad + \{F, L_J\}\frac{\partial H}{\partial L_J}$

Read off Poisson brackets:

$\{R_{iI}, R_{iJ}\} = 0$ $\qquad \{R_{iI}, L_J\} = -\varepsilon_{IJK}R_{iK}$

$\hookrightarrow \{L_I, R_{iJ}\} = -\varepsilon_{IJK}R_{iK}$ (ε & PB antisym)

$\{L_I, L_J\} = -\varepsilon_{IJK}L_K$

minus sign important: body frame angular mom. algebra, not space frame.

(2) done! "Momenta" L_I are not canonical.

$T^*SO(3)$ = "right" coordinates are not canonical

(3) Pick Hamiltonian $H \to$ physics.

"Free" rigid body $\to$ left-$SO(3)$ invariance.

Space frame ij indices contract...

$H = H(R_{iI}R_{iJ}, L_I) = H(L_I)$ since $\delta_{IJ} = R_{iI}R_{iJ}$

generalized translation symmetry $\rightsquigarrow H(x,p) = H(p)$

Build H in spirit of Effective theory...

$$H = \cancel{A} + \cancel{B_I L_I} + \underbrace{\frac{1}{2} C_{IJ} L_I L_J} + \cdots$$

A = const. drop out of EOM

$L \to -L$ under time-reversal so TRS $\Rightarrow$ $B = 0$.

diagonalize C_{IJ} (work in e-basis of C) : $H = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$

moments of inertia

Hamilton's equations:

$$\dot{L}_1 = \{L_1, H\} = -\left(L_3 \frac{\partial H}{\partial L_2} - L_2 \frac{\partial H}{\partial L_3}\right) \qquad \text{etc.}$$
$$= -L_3 \frac{L_2}{I_2} + L_2 \frac{L_3}{I_3}$$

Write: $L_I = I_I \omega_I$ (no sum on I):

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\,\omega_2 \omega_3 \quad \ldots \qquad \text{Euler's equations}$$

Symmetry (left-SO(3)) + nontrivial Poisson brackets:

$$\dot{L}_I = \varepsilon_{IJK} L_J \frac{\partial H}{\partial L_K} \longrightarrow \varepsilon_{IJK} L_J \omega_K \quad \text{def.}$$
$$= (\vec{L} \times \vec{\omega})_I$$