PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 26
Hamiltonian field theory

October 25

It was natural to take a continuum limit:

"mechanics" (point particle) $\longrightarrow$ field theory (continuum)

$$S=\int dt\, L(x_i,\dot{x}_i) \longrightarrow S=\int d^Dx\, \mathcal{L}(\phi,\partial_\mu\phi)$$

It usually is NOT as nice to generalize Hamiltonians...

$$S=\int dt\,[p_i\dot{x}_i - H] \longrightarrow \cancel{S=\int dt\, dx\,[\pi\partial_t\phi\,\partial_x\sigma+\cdots-\mathcal{H}]}$$

~~"multi-symplectic" geometry~~

$$\longrightarrow S=\int dt\, d^dx\,[\pi\partial_t\phi-\mathcal{H}]$$

time is still special

E.g. in relativity, Lagrangian field theory had manifest Lorentz invariance, but not Hamiltonian.

Example 1: Taking the continuum limit

cf lec 12: chain of interacting particles (~1d solid)

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i \frac{1}{2} k(\phi_i - \phi_{i+1})^2 \quad \text{w/} \quad \{\phi_i, p_j\} = \delta_{ij}.$$

$$S = \int dt \left[\sum_i p_i \dot{\phi}_i - H(\phi, p)\right] \qquad \int dx\, \mathcal{H} = H$$

$$\hookrightarrow \; S = \int dt\, dx \left[\pi\, \partial_t \phi - \mathcal{H}\right] \quad \text{where } \mathcal{H} = \frac{\pi^2}{2\mathcal{M}} + \frac{1}{2}\mathcal{K}(\partial_x\phi)^2$$

"momentum density"

Euler-Lagrange equations:

$$\frac{\delta S}{\delta \pi} = 0 = \partial_t \phi - \frac{\pi}{\mathcal{M}}, \qquad \frac{\delta S}{\delta \phi} = 0 = -\partial_t \pi - \partial_x(-\mathcal{K}\partial_x \phi)$$

$$\pi = \mathcal{M}\,\partial_t \phi \qquad\qquad \downarrow \qquad \partial_t \pi = \mathcal{K}\partial_x^2 \phi$$

$$\hookrightarrow \; \mathcal{M}\partial_t^2 \phi = \mathcal{K}\partial_x^2 \phi$$ (same wave equation as Lec 12)

Hamiltonian interpretation?

canonically conjugate fields:

$$\{\phi_i, p_j\} = \delta_{ij} \xrightarrow{\text{continuum}} \{\phi(x), \pi(x')\} = \delta(x - x')$$

Dirac δ: $\int_{-\infty}^{\infty} dx\, \delta(x) f(x) = f(0)$.

[e.g. $\sum_j \{\phi_i, p_j\} = 1 \;\sim\; \int dx' \{\phi(x), \pi(x')\} = 1$]

Claim: If $H = \int dx\, \mathcal{H}$ (or more generally), then $\dot{F} = \{F, H\}$.

Technical tool: functional derivative!

$$\frac{d}{dt} F[\phi, \pi] = \int dx \left[\partial_t \phi(x) \frac{\delta F}{\delta \phi(x)} + \partial_t \pi(x) \frac{\delta F}{\delta \pi(x)}\right] \qquad (\text{vs. } \dot{F} = \dot{\xi}^\alpha \partial_\alpha F)$$

chain rule

Plug in Hamilton's equations:

$$S = \int dt\, dx\, (\pi \partial_t \phi) - \int dt\, H[\pi, \phi]$$

$$\underbrace{\frac{\delta S}{\delta \pi(x,t)}}_{\text{varying over field in both } x \text{ \& } t} = \partial_t \phi - \underbrace{\frac{\delta H}{\delta \pi(x)}}_{\text{functional derivative on } H \text{ as if } H[\phi(x), \pi(x)]}$$

(generalize $\dot{\phi}_i = \frac{\partial H}{\partial p_i}$)

(i.e. @ fixed t)

Similarly: $\partial_t \pi = -\frac{\delta H}{\delta \phi}$. So:

$$\dot{F} = \int dx \left[\frac{\delta F}{\delta \phi(x)} \frac{\delta H}{\delta \pi(x)} - \frac{\delta F}{\delta \pi(x)} \frac{\delta H}{\delta \phi(x)} \right] \stackrel{\text{definition}}{=} \{F, H\}.$$

generalized Poisson bracket

Example 2: Klein-Gordon theory

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 \quad \leftarrow \text{Lorentz invariance manifest!}$$

$$= \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} \underbrace{\partial_i \phi \partial_i \phi}_{\text{ij index: spatial derivatives only}} - \frac{m^2}{2} \phi^2$$

Lorentz NOT as clear

Legendre transform: $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$

so: $\mathcal{H} = \pi \partial_t \phi - \mathcal{L} = \frac{1}{2} \pi^2 + \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{m^2}{2} \phi^2$

$$\partial_t \phi(x) = \{\phi(x), H\} = \{\phi(x), \int dx' \mathcal{H}\}$$

$$= \int dx' \{\phi(x), \pi(x')\} \frac{\delta H}{\delta \pi(x')} = \int dx' \delta(x - x') \pi(x')$$

$$= \pi(x)$$

$$\partial_t \pi = \{\pi(x), H\} = -\frac{\delta H}{\delta \phi(x)} = -\frac{\partial \mathcal{H}}{\partial \phi} + \partial_i \frac{\partial \mathcal{H}}{\partial(\partial_i \phi)} = \partial_i \partial_i \phi - m^2 \phi$$

↳ Combine: $\partial_t^2 \phi = \partial_x^2 \phi - m^2 \phi$ (Klein-Gordon eq)

Comment: Hamiltonian structure is "forced"?

↳ historical path to QFT.

Example 3: U(1) symmetry / superfluid (HW5/ Lec 14)

$$\mathcal{L} = i\bar{\psi}\partial_t\psi - \frac{1}{2m}\partial_x\bar{\psi}\partial_x\psi - V(\bar{\psi}\psi)$$

$$\hookrightarrow \frac{\delta S}{\delta\bar{\psi}} = 0 = i\partial_t\psi - V'(\bar{\psi}\psi)\psi + \frac{1}{2m}\partial_x^2\psi \leftarrow$$ (nonlinear) Schrödinger equation

Key observation: $\underbrace{\mathcal{L} \text{ only has one } \partial_t}$

Clue: Hamiltonian perspective will be fruitful.

$i\bar{\psi}$ is canonical conjugate of ψ: $\{\psi(x), i\bar{\psi}(x')\} = \delta(x-x')$

$\hookrightarrow$ in QM: become $[\psi(x), \psi^\dagger(x')] = -i \cdot i\,\delta(x-y)$

$i[A,B] \longleftrightarrow \{A,B\}$

$$[\psi(x), \psi^\dagger(x')] = \delta(x-x')$$

which is creation/annihilation algebra of harmonic oscillator

$\hookrightarrow$ "second quantization"

Also: $\mathcal{L}$ has a U(1) symmetry:

$\psi \to \psi e^{i\alpha}$, $\bar{\psi} \to \bar{\psi} e^{-i\alpha}$ for const. α

Follow HW5 and write:

$\psi = \sqrt{\rho}\, e^{-i\theta}$ and $\bar{\psi} = \sqrt{\rho}\, e^{+i\theta}$

$\rho \to \rho$ $\leftarrow$ particle density

$\theta \to \theta - \alpha$

$\hookrightarrow$ a few lines of algebra...

$$\mathcal{L} = \boxed{\rho\,\partial_t\theta} - \frac{\rho}{2m}(\partial_x\theta)^2 - \dots - V(\rho)$$

phase θ and density ρ are canonical conjugate variables:

$$\{\theta(x), \rho(x')\} = \delta(x-x')$$

The U(1) symmetry is generated by $Q = \int dx\, \rho(x)$:

total charge/particle number

$$\theta(x) \longrightarrow \theta(x) + \varepsilon \{\theta(x), Q\}$$

$$\{\theta(x), Q\} = \int dx' \{\theta(x), \rho(x')\} = \int dx\, \delta(x - x') = 1$$

So $\theta(x) \to \theta(x) + \varepsilon$

Noether's Thm still works: symmetry $\Longleftrightarrow$ conservation law.