PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 27
Generating functions of canonical transformations

October 28

Hamiltonian mechanics: $F(x,p) \to \dot{F} = \{F, H\}$.

① algebra of symmetry generators ...
② closest to (conventional) QM
③ insight to solvable (integrable) vs. not solvable (chaotic) systems

rest of class

First goal: what coords on phase space "highlight" solvability

Today: finding good large $\underbrace{\text{canonical transformations}}$.

(x,p) coords on phase space $\xi^\alpha \to \eta^\alpha(\xi)$: $\{\xi^\alpha, \xi^\beta\} = \{\eta^\alpha, \eta^\beta\}$.

(x,p)

Lec 22: infinitesimal CT generated by function F:

$$\xi^\alpha \to \xi^\alpha + \varepsilon\{\xi^\alpha, F\} + \cdots$$

infinitesimal

$$\mathrm{ad}_F\, \xi^\alpha = -\{\xi^\alpha, F\}$$

$$\frac{d\xi^\alpha}{ds} = \{\xi^\alpha, F\} \longrightarrow \xi^\alpha(s) = \eta^\alpha = e^{-s\cdot \mathrm{ad}_F}\xi^\alpha$$

What's F?? Not practical...

lec 24: $\omega_{\alpha\beta} = \partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha$

Idea: $\omega_{\alpha\beta}$ is invariant if $\xi^\alpha \to \eta^\alpha$. Also, PB invariant.

$$\lambda = \lambda_\alpha d\xi^\alpha \leftarrow \lambda_\alpha \qquad \theta_\alpha \to \theta = \theta'_{\alpha'} d\eta^{\alpha'} = \theta_\alpha d\xi^\alpha$$

$$\omega_{\alpha\beta} = \underbrace{\partial_\alpha \lambda_\beta - \partial_\beta \lambda_\alpha = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha}$$

$$0 = \partial_\alpha (\lambda - \theta)_\beta - \partial_\beta (\lambda - \theta)_\alpha$$

Solved by: $(\lambda - \theta)_\alpha = \partial_\alpha F$

A la "thermodynamics":

$$dF = \lambda_\alpha d\xi^\alpha - \theta'_{\alpha'} d\eta^{\alpha'}$$

Now: $\xi^\alpha = (x_i, p_i)$ $\qquad$ $\eta^\alpha = (X_i, P_i)$

$\{x_i, p_j\} = \delta_{ij}$ $\qquad$ $\{X_i, P_j\} = \delta_{ij}$

$\lambda = \lambda_\alpha d\xi^\alpha = p_i dx_i$ $\qquad$ $\theta = P_i dX_i$

$\lambda_{p_i} = 0$ & $\lambda_{x_i} = p_i$

$$dF = \lambda - \theta = p_i dx_i - P_i dX_i$$

Tempting? Suppose $F(x_i, X_i)$. Then

$$dF_1 = \frac{\partial F_1}{\partial x_i} dx_i + \frac{\partial F_1}{\partial X_i} dX_i$$

Therefore: $p_i = \frac{\partial F_1}{\partial x_i}$ $\qquad$ $P_i = -\frac{\partial F_1}{\partial X_i}$

This works <u>if</u> x_i & X_i are all $\underbrace{\text{independent.}}$

each point in phase space corresponds to unique (x_i, X_i)

This is called Type-1 CT, generated by F_1

Example 1: take phase space $\mathbb{R}^2$.

Suppose $X = p - 2\lambda x$. How do we find P?

$$p = \frac{\partial F_1}{\partial x} = X + 2\lambda x$$

Therefore $F_1 = Xx + \lambda x^2 + g(X)$

Now: $P = -\frac{\partial F_1}{\partial X} = -g'(X) - x$. let's take $g = 0$.

Re-write: $(X, P) = (p - 2\lambda x, -x)$.

Check $\{X, P\} = \{p - 2\lambda x, -x\} = -\{p, x\} = 1$. ✓

There are 2 natural choices of $\lambda = \lambda_\alpha d\xi^\alpha$:

(A) $\lambda_\alpha d\xi^\alpha = p_i dx_i$ (B) $\lambda_\alpha d\xi^\alpha = -x_i dp_i$

$$\omega_{\alpha\beta} = \begin{array}{cc} x & p \\ \end{array}$$
$$\omega_{\alpha\beta} = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right)$$

Pick (A) or (B) for old and new coordinates...

$2 \times 2 = 4$ common types of large CTs:

Type 1: $F_1(x, X)$ w/ $dF_1 = p_i dx_i - P_i dX_i$

☆Type 2: $F_2(x, P)$ w/ $dF_2 = p_i dx_i + X_i dP_i$

Type 3: $F_3(p, X)$ w/ $dF_3 = -x_i dp_i - P_i dX_i$

Type 4: $F_4(p, P)$ w/ $dF_4 = -x_i dp_i + X_i dP_i$

Example 2: infinitesimal CTs are type 2

Choose: $F_2 = x_i P_i + \varepsilon F(x_i, P_i) + \cdots$

$$p_i = \frac{\partial F_2}{\partial x_i} = P_i + \varepsilon \frac{\partial F}{\partial x_i} \qquad X_i = x_i + \varepsilon \frac{\partial F}{\partial P_i}$$

In F: $F(x, P) \approx F(x, p) + \mathcal{O}(\varepsilon)$. So re-write:

$$P_i = p_i - \varepsilon \frac{\partial F(x,p)}{\partial x_i} + \cdots \quad \text{and} \quad X_i = x_i + \varepsilon \frac{\partial F(x,p)}{\partial p_i} + \cdots$$

Example 3: going to polar coordinates.

$x = r\cos\theta \qquad y = r\sin\theta$

$$F_2(x, y, P_r, P_\theta) = P_r \cdot r(x,y) + P_\theta \cdot \theta(x,y)$$

such that $r = \frac{\partial F_2}{\partial P_r}$ ✓ etc.

$$F_2 = P_r \sqrt{x^2+y^2} + P_\theta \arctan\frac{y}{x}.$$

More on HW9

Time-dependent CTs:

Consider Type-2 CTs (similar formulas hold for others).

w/ $F_2(x_i, P_i, t)$.

Claim: New Hamiltonian $H'(X_i, P_i, t) = H(x(X,P), p(X,P,t), t) + \frac{\partial F_2}{\partial t}$.

while $dF_2 = p_i dx_i + X_i dP_i$.

Why? general G: $\dot{G} = \partial_t G + \{G, H\}$

Plug-in: $G = X_i = \partial F_2 / \partial P_i$:

$$\dot{X}_i = \partial_t X_i + \{X_i, H\} = \frac{\partial^2 F_2}{\partial t \partial P_i} + \frac{\partial H}{\partial P_i} = \frac{\partial H'}{\partial P_i}.$$

So H' change ensures Hamilton's equations unchanged.