PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 28
Hamilton-Jacobi equation

October 30

lec 27: Type 1 CT: $(x_i, p_i) \longrightarrow (X_i, P_i)$ where
(x_i, X_i) are independent: point in phase space specified uniquely

Generating function: F_1: $dF_1 = p_i dx_i - P_i dX_i$ $\left(p_i = \frac{\partial F(x_i, X_i, t)}{\partial x_i}\right)$

New Hamiltonian $H'(X_i, P_i, t) = \frac{\partial F_1}{\partial t} + H(x(X,P,t), p(X,P,t))$.

Can we choose F_1 so that:
1. $H' = 0$.
   ↳ 2. (X_i, P_i) are constants of motion: $\dot{X}_i = \frac{\partial H'}{\partial P_i} = 0$.

Completely solves the system! (integrable...)

If succesful: each (X_i, P_i) corresponds to choice of initial cond.

then find $x_i(X_i, P_i, t)$ and $p_i(X, P, t)$.

Give this F_1 special name: S: $p_i = \frac{\partial S}{\partial x_i}$

$$H' = 0 = \frac{\partial S}{\partial t} + H\left(x_i, \frac{\partial S}{\partial x_i}, t\right)$$ ← Hamilton-Jacobi equation.

Philosophy: $S(x_i, t)$ ← implicit will be X_i (new coords as const. of motion)

HJ: PDE in $\overset{x_i}{n} + \overset{t}{1}$ dimensions

Ham eq's: $2n$ ODEs for x_i & p_i

why trade ODE for PDEs?

But! Clever solution strategies exist...
...historically HJ used often to establish integrability

Example 1: central force problem.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

↓ Legendre transform

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

↓ HJ:

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + V(r)$$

Not looking for most general solution...
use "separation of variables": additive

$$S(r, \theta, t) = S_r(r) + S_\theta(\theta) + S_t(t)$$

HJ: $$0 = \underbrace{S_t'(t)}_{t} + \underbrace{\frac{1}{2m}(S_r')^2}_{r} + \underbrace{\frac{1}{2mr^2} S_\theta'(\theta)^2}_{\frac{f(\theta)}{r^2}} + \underbrace{V(r)}_{r}$$

↓

only part w/ t-dep: this term is constant:

$$S_t(t) = -Et$$

integration const. [think of X_1]

And: $S_\theta'(\theta) = \text{const.}$ as only 3rd term depends on θ...

$S_\theta(\theta) = \ell\,\theta$ ← integration const. (think of as X_2)

Then: $S = S_r(r) + \ell\theta - Et$ and:

$$E = \frac{1}{2m}(S_r')^2 + \ell^2/(2mr^2) + V(r)$$

"Solved by quadratures":

$$S_r(r) = \int dr \sqrt{2mE - \frac{\ell^2}{r^2} - V(r)}$$

Now: $S(r,\theta,t;E,\ell) = \int dr \sqrt{\cdots} + \ell\theta - Et.$

and $X_1 \to E$, $X_2 \to \ell$

Finish solution by finding 2 more constants $P_{1,2}$:

$$-P_1 = \frac{\partial S}{\partial E} = -t + \int \frac{dr \cdot 2m}{2\sqrt{2mE - \ell^2/r^2 - V(r)}}$$

$$-P_2 = \frac{\partial S}{\partial \ell} = \theta - \int \frac{dr\; 2\ell/r^2}{2\sqrt{2mE - \ell^2/r^2 - V(r)}}$$

trajectories $t(r)$ & $\theta(r)$.

P_1 = "initial time"
P_2 = "initial angle"

Example 2: Particle in magnetic field:

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \dot{y}\,qBx \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qBx$$

$$\hookrightarrow H = \frac{1}{2m}p_x^2 + \frac{1}{2m}(p_y + qBx)^2$$

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y} + qBx\right)^2.$$

Separation of variables:

$$S(x,y,t) = -Et + p_y y + S_x(x)$$

$$2mE = (S_x')^2 + (p_y + qBx)^2$$

$$\hookrightarrow S_x(x) = \int dx \sqrt{2mE - (p_y + qBx)^2}$$

If new $X_{1,2}$: $X_1 = E$ and $X_2 = p_y$:

$$-P_1 = \frac{\partial S}{\partial E} = -t + \int \frac{m\, dx}{\sqrt{2mE - (p_y + qBx)^2}} \rightarrow$$

Let: $p_y + qBx = \sqrt{2mE}\cos\theta$

$qB \cdot dx = -\sqrt{2mE}\sin\theta\, d\theta$

$$-P_2 = \frac{\partial S}{\partial p_y} = y - \int \frac{dx}{\sqrt{2mE - (p_y + qBx)^2}} (p_y + qBx)$$

$$-P_1 = -t + \frac{m}{qB} \int \frac{-\sin\theta\, d\theta}{\sqrt{1 - \cos^2\theta}} = -t + \frac{m}{qB}\theta$$

$$\hookrightarrow \cos\left[(t - P_1)\frac{qB}{m}\right] = \cos\theta = \frac{p_y + qBx}{\sqrt{2mE}}$$

$$-P_2 = y + \frac{\sqrt{2mE}}{qB}\sin\theta \rightsquigarrow (y + P_2)^2 = \frac{2mE}{(qB)^2}(1 - \cos^2\theta)$$

$$\frac{2mE}{(qB)^2} = (y + P_2)^2 + \left(x + \frac{p_y}{qB}\right)^2.$$

Thus: trajectory follows circular motion in plane:
radius/center fixed by 3 int. const.

P_1 eq: move at const. speed

Summary: HJ works well when you can $\underbrace{\text{separate variables}}$
$\hookrightarrow$ integrability.

e.g. HW10 $\begin{cases} - \text{clever coord systems} \\ - \text{special potentials.} \end{cases}$

HJ $\rightsquigarrow$ higher-dim WKB in QM.