PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 29
Action-angle variables

November 1

Lec 27: Type 2 CT from $(x_i, p_i) \to (X_i, P_i)$

assume that (x_i, P_i) are independent coords

$$dF_2 = p_i\, dx_i + X_i\, dP_i$$

gen func

$\hookrightarrow$ $p_i = \frac{\partial F_2}{\partial x_i}$ and $X_i = \frac{\partial F_2}{\partial P_i}$

Goal: clever CT to reveal integrability. (solvability)

w/ t-ind. type 2 CT: $H = H(P_i)$

Then: $\dot{X}_i = \frac{\partial H}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial H}{\partial X_i} = 0$

$\dot{X}_i = \text{const.} = \omega_i$ So $P_i = \text{const.}$

$X_i(t) = X_i(0) + \omega_i(\vec{P})t$ $P_i(t) = P_i(0).$

Assume: H is t-ind. and thus F_2 t-ind.

If we find such nice coords... call them action-angle variables

$X_i \to \phi_A$ (angle) $P_i \to J_A$ (action)

ABC... denote A-A variables.

General, no sum convention on repeated indices

How to find J_A?

Motivation: if H-J solved by separation of variables:

$$S(x_A, t) = -Et + \sum_A W_A(x_A; X)$$

← integration const.

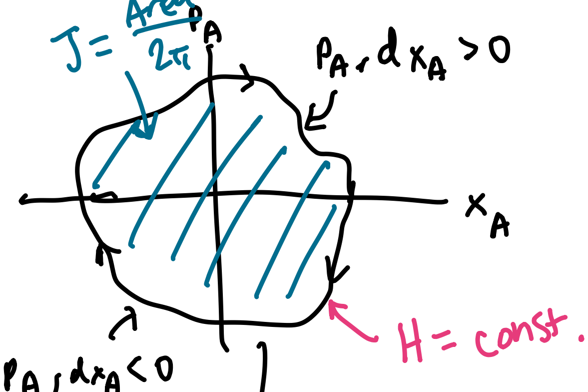
$$p_A = \frac{\partial W_A(x, X)}{\partial x_A}$$

const. of motion !!

Idea to extract X_A: $J_A = \frac{1}{2\pi}\oint p_A \, dx_A$ "action variables"

$$J_A = \frac{1}{2\pi}\oint dx_A p_A = \frac{1}{2\pi}\oint dx_A \frac{\partial W_A}{\partial x_A} \overset{?}{=} \underbrace{\frac{1}{2\pi}\oint dW_A}_{} \cancel{= 0} \; ? \quad \text{no!}$$

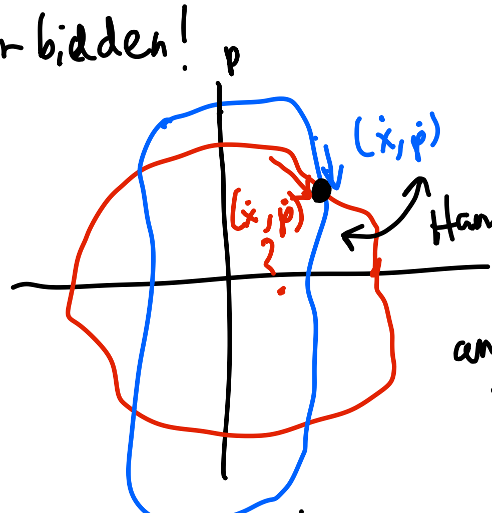
interpret integral as area enclosed in phase space



[Assumption here: trajectory is closed... (can try to generalize...)

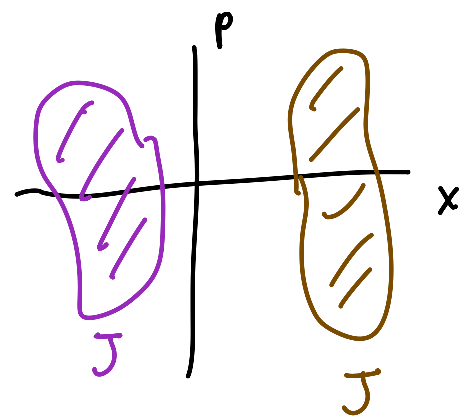
trajectory associated to sol'n of Hamilton's eqs.
→ since stay on same traj for all t, J_A is a const. of motion

For bidden!



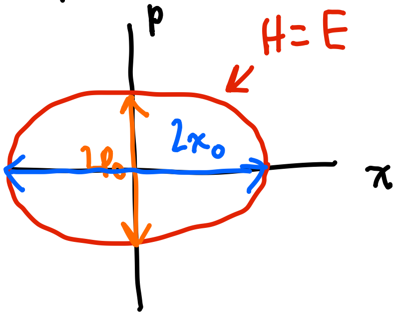
Ham eq is first order, ambiguity impossible

2 trajectories w/ same area?



This case allowed →
AA vars may not be global
(HW 10)

Example: harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$



trajectory = curve w/ const. H.

ellipse

$E = \frac{p_0^2}{2m}$ or $p_0 = \sqrt{2mE}$

$E = \frac{1}{2} m \omega^2 x_0^2$ or $x_0 = \sqrt{\frac{2E}{m\omega^2}}$

Action $J = \frac{\text{Area}}{2\pi} = \frac{\pi x_0 p_0}{2\pi}$ → $\underbrace{\pi}_{\text{(circle area)}} \cdot x_0 \cdot p_0$ ← rescale factor to get to unit circle

$J = \frac{E}{\omega}$

Therefore $H(J) = \omega J$

Hamilton's eqs:

$\dot{J} = -\frac{\partial H}{\partial \phi} = 0$ ✓ $\qquad \dot{\phi} = \frac{\partial H}{\partial J} = \omega$

$J(t) = J(0) \qquad \phi(t) = \phi(0) + \omega t$

$p = p_0 \cos\phi \qquad \longrightarrow \qquad p = \sqrt{2m\omega J} \cos\phi$

$x = x_0 \sin\phi \qquad\qquad\quad x = \sqrt{\frac{2J}{m\omega}} \sin\phi$

Check canonical:

$$\{x, p\} = \sqrt{\frac{2}{m\omega}} \sqrt{2m\omega} \left\{ \sqrt{J} \sin\phi, \sqrt{J}\cos\phi \right\}$$

$$= 2 \left[\frac{\partial(\sqrt{J}\sin\phi)}{\partial\phi} \frac{\partial(\sqrt{J}\cos\phi)}{\partial J} - \frac{\partial(\sqrt{J}\sin\phi)}{\partial J} \frac{\partial(\sqrt{J}\cos\phi)}{\partial\phi} \right]$$

$$= \frac{\sqrt{J}}{\sqrt{J}} \left[\cos^2\phi + \sin^2\phi \right] = 1 \quad !!$$

Usually, goal is just J.

Note $\phi \sim \phi + 2\pi$ is periodic.

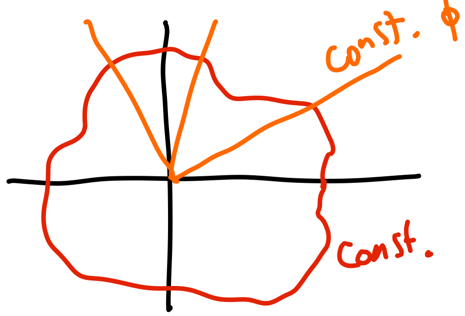
Recap: action - angle variables are natural canonical conjugate coords for integrable systems...

$H = H(J_A)$ so $\dot{J}_A = -\frac{\partial H}{\partial \phi_A} = 0$ $\qquad$ $\dot{\phi}_A = \frac{\partial H}{\partial J_A} = \text{const.} = \omega_A$

n const. of motion

So $\phi_A(t) = \phi_A(0) + \omega_A t$

AA is usually better than HJ?



- J have nice phase space interpretation
- when orbits closed, $\phi_A \sim \phi_A + 2\pi$

nice geometric picture of phase space (Lec 31, ...)

Connection to QM: Bohr-Sommerfeld quantization

→ find AA variables...

conjecture: $J_A = n_A \hbar$ $\qquad n_A = 1, 2, 3, \ldots$

discrete energy levels: $E_{n_1 \ldots n_k} = H(n_1\hbar, n_2\hbar, \ldots, n_k\hbar)$

in 2k-dim phase space

↳ "derive" via WKB approximation up to $n_A \to n_A - \frac{1}{2}$