PHYS 5210
Graduate Classical Mechanics
Fall 2024

Lecture 3
Noether's Theorem

August 30

Symmetry = a coordinate tranrform that leaves EOM invariant: $(x_i, t) \to (\tilde{x}_i(x,t), \tilde{t}(x,t))$

Continuous symmetry generated by

$$\tilde{x}_i = x_i + \varepsilon X_i(x,t) + \cdots \qquad \tilde{t} = t + \varepsilon T(x,t) + \cdots$$

$$0 = \dot{\Phi} + \dot{T}L + T\frac{\partial L}{\partial t} + X_i \frac{\partial L}{\partial x_i} + \frac{\partial L}{\partial \dot{x}_i}(\dot{X}_i - \dot{x}_i \dot{T})$$

$L \to L + \varepsilon \dot{\Phi}$ is physically equivalent (until QM)

<u>Noether's Thm:</u> continuous symmetry $\Rightarrow$ conservation law.

"Noether charge" $\quad Q = T\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) + X_i \frac{\partial L}{\partial \dot{x}_i} + \Phi$

On physical trajectories $\dot{Q} = 0$.

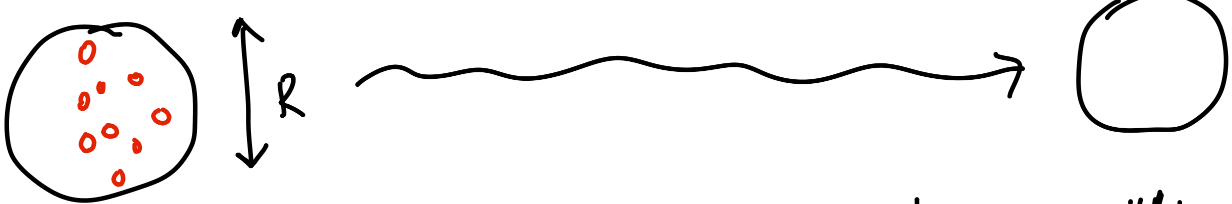
Recall: $X_i \frac{\partial L}{\partial \dot{x}_i} = \sum_{i=1}^{n} X_i \frac{\partial L}{\partial \dot{x}_i}$

Proof:
$$\frac{dQ}{dt} = \frac{d\Phi}{dt} + \dot{T}\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) + T\left(\frac{\partial L}{\partial t} + \cancel{\dot{x}_i \frac{\partial L}{\partial x_i}} + \cancel{\ddot{x}_i \frac{\partial L}{\partial \dot{x}_i}} - \cancel{\ddot{x}_i \frac{\partial L}{\partial \dot{x}_i}} - \cancel{\dot{x}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}}\right) + \dot{X}_i \frac{\partial L}{\partial \dot{x}_i} + X_i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0.$$

Remember: physical traj: $\frac{\delta S}{\delta x_i} = 0 \rightsquigarrow \frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$

Key: symmetry "easily" implemented in Lagrangian mech...
AND Noether's Thm: symmetry $\Rightarrow$ conservation law
right language for effective theory.

Example 1: "Ball" in "empty space"



$\sim 10^{26}$ atoms

1) describe large length/t
2) $\Delta x \gg R$, $\Delta t \gg$ vibration
3) configuration space $x \in \mathbb{R}$

4) postulate: translation symmetry $(X, T) = (1, 0)$
"most general" L: $L(\dot{x}, t) \rightsquigarrow f(\dot{x})$
lec 2

Noether's Thm: $p = \frac{\partial L}{\partial \dot{x}} = f'(\dot{x})$: define as momentum.

Euler-Lag: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = \dot{p} \Rightarrow p = \text{const.}$

- Study motion "near $\dot{x} = v_0$."
Given knowledge of state, $x(t) = v_0 t + z(t)$ small!

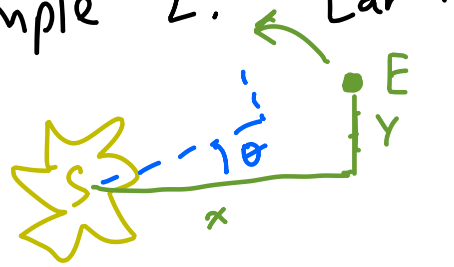
$L = f(v_0 + \dot{z}) = \cancel{f(v_0)} + \cancel{f'(v_0)}\,\dot{z} + \frac{1}{2}f''(v_0)\,\dot{z}^2 + \ldots$

const. const. $\neq 0$ in general total derivative

Simplest choice: $L = \frac{1}{2}m\dot{z}^2 + \ldots$ $f''(v_0) \to m$ "mass"

$p = \frac{\partial L}{\partial \dot{x}} \rightsquigarrow m\dot{z}$ | EOM: $m\ddot{z} = 0$ ("Newton")

Example 2: Earth around Sun



1, 2) interested in overall "macroscopic" Earth orbit

3) Config space: $\mathbb{R}^2$ (x, y)

4) No translation... but rotational symmetry

(Plus: time-reversal $\tilde{t} = -t$)

$$\tilde{t} = t \qquad \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

Since rotation is continuous... find generator:

$$\begin{pmatrix}X\\ Y\end{pmatrix} = \frac{d}{d\theta}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \longrightarrow \left.\begin{matrix}X = -y\\ Y = x\end{matrix}\right\}.$$

Find "invariant building blocks"

- length of a vector in $\mathbb{R}^2$

$$\vec{x}\cdot\vec{x} = x^2 + y^2 \qquad \dot{x}^2 + \dot{y}^2 = \dot{\vec{x}}\cdot\dot{\vec{x}}$$

- angle btwn 2 vectors:

$$\vec{x}\cdot\dot{\vec{x}} = x\dot{x} + y\dot{y} \qquad \text{"}\vec{x}\times\dot{\vec{x}}\text{"} = x\dot{y} - y\dot{x}$$

... posit most general Lagrangian: (assume $\Phi = 0$)

$$L(x^2 + y^2,\ \dot{x}^2 + \dot{y}^2,\ x\dot{x} + y\dot{y},\ x\dot{y} - y\dot{x})$$

Noether's Thm:

$$(Q=)\ M = X\frac{\partial L}{\partial \dot{x}} + Y\frac{\partial L}{\partial \dot{y}} = x\frac{\partial L}{\partial \dot{y}} - y\frac{\partial L}{\partial \dot{x}} = \text{constant on EOM}$$

angular momentum

$= xp_y - yp_x$ ($p_{x,y}$ NOT constants)

Switch to polar coordinates: $x = r\cos\theta$
$y = r\sin\theta$

Rotational sym: $\theta \to \theta + \text{const.}$ (translation in θ)

Find $L = L(r^2, \dot{r}^2 + r^2\dot{\theta}^2, \cancel{2r\dot{r}}, \cancel{r^2\dot{\theta}})$

drop for easy ... NOT good idea

Slow motion: Series expand in $\dot{r}^2 + r^2\dot{\theta}^2$:

$$L \approx \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) + \cdots$$

m = "mass"

"potential energy"

(time-reversal $\Rightarrow$ even $\frac{d}{dt}$)

Angular momentum: $M = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{const.}$

$$\frac{\delta S}{\delta r} = 0 = -\frac{dV}{dr} + mr\dot{\theta}^2 - m\ddot{r}$$

$$= -\frac{d}{dr}\left(V(r) + \frac{M^2}{2mr^2}\right) - m\ddot{r}$$

OK b/c EOM, NOT Lagrangian

"Interesting" config space is r only.

"Central force problem"